Higher Invariants in Noncommutative Geometry

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Based on joint work with Sherry Gong, Jianchao Wu, Shmuel Weinberger, Zhizhang Xie, Xiaoman Chen, and Jinmin Wang, Hang Wang, Hongzhi Liu, Hao Guo, Rudolf Zeidler, From the perspective of Alain Connes' noncommutative geometry, the Dirac operator D encodes all geometric data of a spin Riemannian manifold:

$$d(x, y) = sup\{|f(x) - f(y)| : ||[D, f]|| \leq 1\}.$$

Dirac Operator

Looking for a first order differential operator on the 2-d Euclidean space R^2:

$$D = c_1 \frac{\partial}{\partial x_1} + c_2 \frac{\partial}{\partial x_2}$$

satisfying

$$D^2 = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}.$$

$$D^2 = c_1^2 \frac{\partial^2}{\partial x_1^2} + c_1 c_2 \frac{\partial^2}{\partial x_1 \partial x_2} c_1^2 + c_2 c_1 \frac{\partial^2}{\partial x_2 \partial x_1} + c_2^2 \frac{\partial^2}{\partial x_2^2}.$$

Using $\frac{\partial^2}{\partial x_1 \partial x_2} = \frac{\partial^2}{\partial x_2 \partial x_1}$,

$$c_1^2 = -1, \quad c_1 c_2 + c_2 c_1 = 0, \quad c_2^2 = -1.$$

Dirac Operator

Let

$$c_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$c_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

c_1 and c_2 satisfy the equations

$$c_1^2 = -1, \quad c_1 c_2 + c_2 c_1 = 0, \quad c_2^2 = -1.$$

Dirac Operator on R^n

Hamilton and Clifford:

$$D = c_1 \frac{\partial}{\partial x_1} + \cdots + c_n \frac{\partial}{\partial x_n},$$

where

$$c_i^2 = -1, \quad c_i c_j + c_j c_i = 0$$

when $i = j$.
We have

$$D^2 = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}.$$

Dirac operator on a manifold

$$D = c_1\nabla_1 + \cdots + c_n\nabla_n.$$

Question: Is D^2 equal to the Laplacian?

Dirac operator and scalar curvature

Lichnerowicz formula

$D^2 = Laplacian + \frac{k}{4}$, where k is scalar curvature.

Corollary

If the scalar curvature $k > 0$, then D is invertible.

Fredholm index theory

If D is the Dirac operator on a compact manifold, then D is Fredholm. In particular, $Kernel(D)$ and $Kernel(D^*)$ are finite dimensional.

Definition

The Fredholm index of D is defined to be:

$$index(D) = dim\ Kernel(D) - dim\ Kernel(D^*).$$

The Fredholm index is an obstruction to invertibility and invariant under small perturbation.

Theorem (Atiyah and Singer)

If D is the Dirac operator on a compact spin manifold, then $index(D) = \hat{A}(M)$.

Here $\hat{A}(M)$ is the A-hat genus of M, a topological invariant.

Corollary: If a compact manifold M has a Riemannian metric with positive scalar curvature, then $\hat{A}(M) = 0$.

Positive Scalar Curvature for Higher Dimensional Torus

Question: Does the torus T^n have positive scalar curvature?

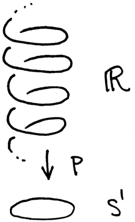
Obsevation: $\hat{A}(T^n) = 0$. Hence the Atiyah-Singer index theorem doesn't apply here.

Answer: No.

Schoen-Yau, Gromov-Lawson.

Symmetries

Taking advantage of symmetries: $T^n = R^n/Z^n$.



Higher Index Theory

Lift the Dirac operator D to the Dirac operator $\tilde{D}$ on R^n. More generally let G be the fundamental group of a compact Riemmanian manifold X and let $M = \tilde{X}$.
Let T be a kernel operator acting on $L^2(M)$ by

$$(Tf)(x) = \int_M k(x,y) f(y) dy.$$

The propagation of T is defined to be $sup\{d(x,y) : k(x,y) \neq 0\}$.

Definition

(1) The Roe algebra $C^*(M)$ is the operator norm closure of all kernel operators with finite propagation.
(2) The equivariant Roe algebra $C^*(M)^G$ is the operator norm closure of all G-invariant kernel operators with finite propagation.

The equivariant Roe algebra $C^*(M)^G$ is Morita equivalent to the reduced group $C_r^*(G)$.

Higher Index Theory (continued)

D is invertible modulo the Roe algebra $C^*(M)$ (equivariant Roe algebra $C^*(M)^G$) and hence we can define a higher index $Index(\tilde{D})$ in $K_*(C^*(M))$ or $K_*(C^*(M)^G)$.

Let

$$F = \frac{D}{\sqrt{1+D^2}}.$$

Defintion of higher index

When the dimension of M is odd, we define the higher index of D by:

$$Index(D) = exp(2\pi i \frac{F+1}{2}) \in K_1(C^*(M)^G).$$

Higher index of D is an obstruction to invertibility of D and invariant under small perturbation..

Higher Index Theory (continued)

Theorem

The higher index of the Dirac operator $\tilde{D}$ on R^n is non-zero.

Corollary: The higher dimensional torus T^n can not have a Riemannian metric with positive scalar curvature.

Localization algebras

Definition

Let X be a compact manifold with $\pi_1(X) = G$ and $M = \tilde{X}$. The localization algebra $C_L^*(M)^G$ is the operator norm closure of

$$\{f : [0, \infty) \to C^*(M)^G\}$$

where f is uniformly bounded and uniformly continuous function such that

$$propagation(f(t)) \to 0 \quad as \quad t \to \infty.$$

Let $e : C_L^*(M)^G \to C^*(M)^G$ be the evaluation homomorphism: $e(f) = f(0)$.

Baum-Connes map

The evaluation homormorphism e induces the Baum-Connes map:

$$e_* : K_*(C_L^*(M)^G) \to K_*(C^*(M)^G).$$

Localization algebras and the Dirac operator

Let D be the Dirac operator on M. We normalize D as follows:

$$F = \frac{D}{\sqrt{1+D^2}}.$$

For each positive integer n, let $\{U_k^{(n)}\}_k$ be a G-equivariant open cover of M such that $diam(U_k^{(n)}) < \frac{1}{n}$. Let $\{\phi_k^{(n)}\}$ be a partition of unity subordinate to the open cover. We define F_n by:

$$F_n = \sum_k \sqrt{\phi_k^{(n)}} F \sqrt{\phi_k^{(n)}}.$$

Let $F_0 = F$. Define the localized operator

$$F_t = (t-k+1)F_k + (t-k)F_{k+1}$$

for all $t \in [k, k+1]$ and $k \geq 0$. We have

$$propagation(F_t) \to 0$$

as $t \to \infty$.

Local higher index of the Dirac operator

$F_\bullet$ is invertible modulo the localization algebra $C_L^*(M)^G$. Hence we can define the local higher index $Index_L(D) \in K_*(C_L^*(M)^G)$.

Defintion of local higher index

When the dimension of M is odd, we define the higher index of D by:

$$Index_L(D) = exp(2\pi i \frac{F_\bullet + 1}{2}) \in K_1(C_L^*(M)^G).$$

We have

$$e_*(Index_L(D)) = Index(D).$$

Higher Index Theorem for Aspherical Manifolds

Definition: A compact manifold X is called aspherical if its universal cover $\tilde{X}$ is contractible.

The following local-global principle provide an algorithm of determining when the higher index is nonvanishing.

Strong Novikov conjecture

If X is aspherical and $M = \tilde{X}$, then the Baum-Connes map

$$e_* : K_*(C_L^*(M)^G) \to K_*(C^*(M)^G)$$

is rationally injective.

Corollary (Rosenberg): If X is aspherical, then X can not have a Riemannian metric with positive scalar curvature.

Higher Index Theorem for Aspherical Manifolds

Connes, Moscovici, Kasparov, Higson, Skandalis, Lafforgue, Gromov, Lustig, Mischenko, Weinberger, Tu, Mathai, ...

Theorem (Alain Connes)

The strong Novikov conjecture holds for Gelfand-Fuchs classes of the diffeomorhism group.

Connes: Cyclic cohomology and the transverse fundamental class of a foliation, 1986.

Higher Index Theorem for Aspherical Manifolds

Theorem (Sherry Gong, Jianchao Wu, Yu)

If M is aspherical and the fundamental group of M is a discrete subgroup of a diffeomophism group of a smooth compact manifold, then the rational strong Novikov conjecture is true, i.e. the higher index of the Dirac operator $\tilde{D}$ on $\tilde{M}$ is non-zero.

Open problem: Remove the discreteness condition in the above theorem.

Locality of Differential Operators

When a differential operator D is local in the sense, if $f \in C^\infty(M)$, then

$$support(Df) \subset support(f).$$

Here $support(f) = \{x : f(x) \neq 0\}$.

Secondary Invariants of Differential Operators

When a differential operator D is invertible, then its higher index is 0. In this case, a natural secondary invariant of D arises. This secondary invariant is an obstruction for its inverse to be local.

Example: when M has positive scalar curvature, the Dirac operator is invertible.

Example: (Hilsum-Skandalis, Zenobi). Let X be a compact manifold. If there exists another compact manifold N and a homotopy equivalence $f : N \to X$, we can define a relative signature operator:

$$d_f = \begin{bmatrix} d_{\tilde{N}} & f^* \\ 0 & d_{\tilde{X}} \end{bmatrix}, \qquad D_f = d_f + d_f^*$$

The assumption that f is a homotopy equivalence implies that D is invertible. The secondary invariant of D_f measures how far f is from a homeomophism.

Secondary Invariants of Differential Operators

When the differential operator D is invertible, the higher index of D is trivial. Let $h(s)$ be a trivialization of the higher index $Index(D)$, i.e. $h(0)$ is trivial and $h(1) = Index(D)$. We put together this trivialization h with the local higher index to define to secondary invariant.

Higher rho invariant (Higson-Roe)

Let $\rho_t(D) = h(t)$ when $t \in [0,1]$ and $\rho_t(D) = (Index_L(D))(t-1)$ when $t \in [1, \infty)$. We define the higher rho invariant of D to be the K-theory class

$$[\rho.(D)] \in K_*(C^*_{L,0}(M)^G),$$

where

$$C^*_{L,0}(M)^G) = \{f \in C^*_{L,0}(M)^G) : f(0) = 0\}.$$

If M is a manifold with boundary ∂M and the boundary has positive scalar curvature, then the higher rho invariant of $D_{\partial M}$ is the boundary term of the higher index of the Dirac operator D_M (Piazza-Schick, Xie-Yu).

Space of positive scalar curvature metrics

Application: if g_0 and g_1 are in the same connected component of the space of all positive scalar curvature metrics, then $[\rho(D_{g_0})] = [\rho(D_{g_1})]$.

Let $P(X)$ be the abelian group generated by all positive scalar curvature metrics (Stolz).

Theorem (Piazza and Schick)

Let M be a closed spin manifold with $\pi_1(X) = G$ and $dim\ X = 2k + 1 \geq 5$, which carries a positive scalar curvature metric. Then the rank of $P(X)$ is ≥ 1.

If g is a positive scalar curvature Riemannian metric on a compact spin manifold X and ϕ is a diffeomorphism of X, then $\phi^* g$ is a positive scalar curvature metric.

Define $\tilde{P}(X) = P(X)/P_0(X)$, where $P_0(X)$ is the subgroup of $P(M)$ generated by elements of the form $[g] - [\phi^* g]$ for all $[g] \in P(M)$ and all $\phi \in Diff(X)$.

Theorem (Xie and Yu)

Let M be a closed spin manifold with $\pi_1(X) = G$ and $dim\ X = 2k+1 \geq 5$, which carries a positive scalar curvature metric. If G is residually finite, then the rank of the abelian group $\tilde{P}(X)$ is $\geq N_{fin}(G)$, where

$$N_{fin}(G) = \#\{d \in N : \exists \gamma \in G\ s.t.\ order(\gamma) = d, \gamma \neq e\}.$$

Secondary Invariants

Open Problem

Compute $\rho(D_g) - \rho(D_{\phi^* g})$, where ϕ is a diffeomorphism.

Delocalized traces and secondary invariants

Let $h \in G$ and $h \neq e$. We define the delocalized trace of a kernel operator A by:

$$tr_h(A) = \sum_{g \in <h>} \int_F A(x, gx)dx.$$

The delocalized eta invariant of Lott is defined as:

$$\eta_{<h>}(D) = \frac{2}{\sqrt{\pi}} \int_0^\infty tr_h(De^{-t^2D^2}).$$

Theorem (Xie and Yu)

If the conjugacy class of $h \in G$ has polynomial growth, then

$$\tau_h(\rho(D)) = -\frac{1}{2}\eta_{<h>}(D).$$

This result has been extended to non-compact manifolds with positive scalar curvature at infinity by X. Chen, H. Liu, H. Wang and Yu.

Corollary

If the rational Baum-Connes conjecture holds for G, and the conjugacy class $< h >$ of a non-identity element $h \in G$ has polynomial growth, then the delocalized eta invariant $\eta_{<h>}(D)$ is an algebraic number. If in addition h has infinite order, then $\eta_{<h>}(D)$ vanishes.

This theorem follows from the previous theorem and Lefschetz fixed point theorem of B.-L. Wang and H. Wang.

When G is torsion-free and satisfies the Baum-Connes conjecture, and the conjugacy class $< h >$ of a non-identity element h has polynomial growth, Piazza and Schick have proved the vanishing of $\eta_{<h>}(D)$ by a different method.

In light of the algebraicity theorem, we propose the following question.

Question

What values can delocalized eta invariants take in general? Are they always algebraic numbers?

If the delocalized eta invariant is non-algebraic, the we have a counter example to the Baum-Connes conjecture.

Theorem (Chen, Wang, Xie and Yu)

The pairing of the higher rho invariant with delocalized cyclic cocycles of the group algebra CG can be expressed in terms of Lott's higher eta invariant.

Pizza, Schick and Zenobi have a different approach to the computation of Connes-Chern character of higher rho invariant. It would be interesting to compare the results of these two different approaches.

Higher Atiyah-Patodi-Singer index theory: Leichtnam-Piazza, Gorokhovsky-Moriyoshi-Piazza, P. Hochs-B. Wang-H. Wang, Dai-Zhang.

Work of Deeley-Goffeng, Wahl.

Secondary Invariants of Differential Operators (Additivity)

Let X be a compact topological manifolds. Let $S(X)$ be the abelian group of all homotopy equivalence $f : N \to X$, where N is a compact manifold.

Theorem (Weinberger, Xie and Yu)

The higher rho invariant is additive from $S(X)$ to $K_*(C^*_{L,0}(\tilde{X})^G)$.

Proof: A combination of topology, analysis, and algebra. (Communications in Pure Applied and Mathematics, 2020).

Corollary: There are compact manifolds M whose the topological structure group of S(M) is infinitely generated.

Computation of secondary invariants

Theorem (Guo, Xie and Yu)

The higher rho invariant is multiplicative.

Approximation Theorem (J. Wang, Xie and Yu)

When G is residually finite, secondary invariants can be approximated by their analogues on finite covers if the Baum-Connes conjecture is true.

Open Question

Open Question

Find a formula of $\rho(D_f)$.

Thank you!