Crystallization of C^*-algebras

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Equilibrium states

Let A be a C*-algebra with a time evolution $\sigma\colon \mathbb{R} \to \mathrm{Aut}(\mathbb{R})$. Take a state φ on A.

The state φ is a σ-**KMS state** at the inverse temperature $\beta = \dfrac{1}{T}$ if

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$$

for all σ-analytic a and b.

The state φ is a σ-**ground state** if the function

$$z \mapsto \varphi(b\sigma_z(a))$$

is bounded (hence bounded by $\|a\|\,\|b\|$) in the upper half-plane for all σ-analytic a and b.

A simple motivating example

Consider the Toeplitz algebra $\mathcal{T}$ generated by the shift $u\colon \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$, and the gauge action

$$\sigma_t(u) = e^{it}u.$$

For every $\beta > 0$ there is a unique KMS-state at β (**Gibbs state**)

$$\varphi = (1 - e^{-\beta})^{-1}\,\mathrm{Tr}(\cdot\, e^{-\beta H}), \quad H\delta_n = n\delta_n.$$

There is also a unique ground state

$$\varphi = (\cdot\,\delta_0, \delta_0).$$

We also have $\mathcal{T} \sim_{KK} \mathbb{C}$. Is there anything to this example?

Connes: Can A be *cooled down* to get a simpler algebra retaining some information about A?

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Similar ideas

Crystal bases in representation theory (Kashiwara, Lusztig):

$$\text{Semisimple Lie algebra } \mathfrak{g} \rightsquigarrow U_q\mathfrak{g}\ (q = e^{-h}) \rightsquigarrow \text{let } q = 0$$

Quantum homogeneous spaces as graph algebras: the graph is easier to find for $q = 0$ (Hong–Szymanski)

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Crystal

For $\lambda \in \mathbb{R}$, define

$$A_\lambda = \{a \in A \mid \sigma_t(a) = e^{it\lambda} a \text{ for all } t \in \mathbb{R}\}.$$

We assume that the dynamics σ is almost periodic, that is, the $*$-subalgebra $\mathcal{A}$ of A spanned by the spaces A_λ, $\lambda \in \mathbb{R}$, is dense in A.

Denote by I_λ the ideal $\overline{A_\lambda A_\lambda^*}$ in A_0, and let

$$I = \overline{\sum_{\lambda > 0} I_\lambda}.$$

Definition

We call the C*-algebra $A_c = A_0/I$ the **crystal** of (A, σ).

Ground states

Let $\Gamma \subset \mathbb{R}$ be the additive subgroup generated by the numbers $\lambda \in \mathbb{R}$ such that $A_\lambda \neq 0$. We view Γ as a discrete group.

We have a conditional expectation

$$E\colon A \to A_0, \quad E(a) = \int_{\hat{\Gamma}} \sigma_\chi(a)\, d\chi,$$

so that $E(A_\lambda) = 0$ for $\lambda \neq 0$. We then get a contractive completely positive map

$$\vartheta\colon A \to A_c = A_0/I, \quad \vartheta(a) = E(a) + I.$$

Proposition

If $A_c \neq 0$, then the map $\psi \mapsto \psi \circ \vartheta$ defines a bijection between the state space of A_c and the σ-ground states on A. If $A_c = 0$, then there are no σ-ground states on A.

Fock module

For every $\lambda \in \mathbb{R}$ we can view A_λ as a right C*-Hilbert A_0-module with the inner product $\langle a, b\rangle = a^* b$. Consider the right C*-Hilbert A_c-module

$$X_\lambda = A_\lambda / \overline{A_\lambda I} = A_\lambda \otimes_{A_0} (A_0/I).$$

We have $X_0 = A_c$ and $X_\lambda = 0$ for all $\lambda < 0$. Denote by $\Gamma_+ \subset \Gamma$ the submonoid generated by all $\lambda \geq 0$ such that $X_\lambda \neq 0$.

Definition

The right C*-Hilbert A_c-module

$$\mathcal{F} = \bigoplus_{\lambda \in \Gamma_+} X_\lambda = \Big(\bigoplus_{\lambda \in \Gamma} A_\lambda\Big) \otimes_{A_0} (A_0/I)$$

is called the **Fock module**.

The left action of A on itself defines a $*$-homomorphism $\Lambda \colon A \to \mathcal{L}(\mathcal{F})$.

Vacuum representations

Definition

A representation $\pi\colon A \to B(H)$ is called a **vacuum representation** if $\pi(A)H_0$ is dense in H, where

$$H_0 = \{\Omega \in H \mid \pi(a)^*\Omega = 0 \text{ for all } a \in A_\lambda \text{ and } \lambda > 0\}$$

is the subspace of **vacuum vectors**.

Proposition

A representation $\pi\colon A \to B(H)$ is a vacuum representation if and only if it is induced from a representation ρ of A_c by the Fock module. Furthermore, the map $\rho \mapsto \operatorname{Ind}\rho$ defines an equivalence between the category of representations of A_c and the category of vacuum representations of A.

Corollary

We have:

(i) $A_c = 0$ *if and only if A has no nonzero vacuum representations;*

(ii) $A_c = \mathbb{C}$ *if and only if A has a unique up to equivalence nonzero representation with a cyclic vacuum vector;*

(iii) $A_c \cong K(H)$ *for a nonzero Hilbert space H if and only if A has a unique up to equivalence irreducible vacuum representation.*

Example: Consider the CAR-algebra $A =$CAR(H) with generators $a^*(\xi)$ and $a(\xi) = a^*(\xi)^*$, $\xi \in H$,

$$a^*(\xi)a^*(\zeta) + a^*(\zeta)a^*(\xi) = 0, \quad a(\xi)a^*(\zeta) + a^*(\zeta)a(\xi) = (\zeta, \xi)1,$$

and the gauge action $\sigma_t(a^*(\xi)) = e^{it}a^*(\xi)$. Then $A_c = \mathbb{C}$ and $\mathcal{F}$ is the usual fermionic Fock space.

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Decomposition of states

Definition

A positive linear functional φ on A is said to be of **finite type**, if the corresponding GNS-representation is a vacuum representation. It is said to be of **infinite type** if the corresponding GNS-representation has no nonzero vacuum vectors.

It can be shown show that φ is of finite type if and only if $\varphi = \psi \circ \Lambda$ for a (unique) strictly continuous positive linear functional ψ on $\Lambda(A) \subset \mathcal{L}(\mathcal{F})$, and it is of infinite type if and only if there is no nonzero strictly continuous positive linear functional ψ on $\Lambda(A)$ such that $\varphi \geq \psi \circ \Lambda$.

Proposition

Every positive linear functional φ on A has a unique decomposition $\varphi = \varphi_f + \varphi_\infty$ where φ_f is of finite type and φ_∞ is of infinite type. If φ satisfies the σ-KMS$_\beta$-condition for some $\beta \in \mathbb{R}$, then the same is true for φ_f and φ_∞.

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Induced traces and KMS weights

If B is a C*-algebra, Y is a right C*-Hilbert B-module and τ is a finite positive trace on B, then there is a unique strictly lower semicontinuous, in general infinite, trace $\mathrm{Tr}_\tau = \mathrm{Tr}^Y_\tau$ on $\mathcal{L}(Y)$ such that

$$\mathrm{Tr}_\tau(\theta_{\xi,\xi}) = \tau(\langle \xi, \xi \rangle) \ \text{ for all } \ \xi \in Y,$$

Explicitly, if $(u_i)_{i\in I}$ is an approximate unit in $\mathcal{K}(Y)$ such that, for every i, $u_i = \sum_{\xi \in J_i} \theta_{\xi,\xi}$ for some finite set $J_i \subset Y$, then

$$\mathrm{Tr}_\tau(T) = \lim_i \sum_{\xi \in J_i} \tau(\langle \xi, T\xi \rangle) \ \text{ for } \ T \in \mathcal{L}(Y)_+.$$

More generally (Laca–N), if B is equipped with a time evolution and Y is full and $\mathbb{R}$-equivariant, then, for any inverse temperature β, we have a one-to-one correspondence between the KMS$_\beta$ weights on $\mathcal{K}(Y)$ and B. Any such weight on $\mathcal{K}(Y)$ extends uniquely to a strictly lower semicontinuous weight on $\mathcal{L}(Y)$.

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Von Neumann dimension

We will write $\dim_\tau Y$ for $\mathrm{Tr}_\tau^Y(1)$.

If $(H_\tau, \pi_\tau, \xi_\tau)$ is the GNS-triple associated with τ and $M = \pi_\tau(B)''$, then $\dim_\tau Y$ is the von Neumann dimension of the right Hilbert M-module

$$Y \otimes_B H_\tau$$

with respect to the trace $(\cdot\, \xi_\tau, \xi_\tau)$ on M.

If τ is a tracial state and Y is topologically generated by n elements, then

$$\dim_\tau Y \le n.$$

Classification of KMS states of finite type

Let D be the generator of the dynamics on the Fock module:

$$Dx = \lambda x \quad \text{for} \quad x \in X_\lambda,\ \lambda \in \Gamma_+.$$

Theorem

For every $\beta \in \mathbb{R}$, the map

$$\tau \mapsto \mathrm{Tr}_\tau^{\mathcal{F}}(\Lambda(\cdot)e^{-\beta D})$$

is an affine bijection between the positive traces τ on A_c such that $\sum_{\lambda\in\Gamma_+} e^{-\beta\lambda}\dim_\tau X_\lambda = 1$ and the σ-KMS$_\beta$-states on A of finite type.

Key point: $\Lambda(A)$ is strictly dense in $\mathcal{L}(\mathcal{F})$, hence a KMS state of finite type on A defines a KMS-state on $\mathcal{K}(\mathcal{F})$.

For a tracial state τ on A_0, define a semi-norm $\|\cdot\|_\tau$ on A_λ, $\lambda \in \mathbb{R}$, by

$$\|a\|_\tau = \tau(a^*a)^{1/2}.$$

Theorem

Assume that there exist closed right A_0-submodules $B_\lambda \subset A_\lambda$, $\lambda > 0$, and a number $\beta_0 \geq 0$ such that for every $\beta > \beta_0$ and every tracial state τ on A_0 we have:

(1) $\sum_{\lambda>0} B_\lambda A_{\mu-\lambda}$ *is dense in* A_μ *with respect to the semi-norm* $\|\cdot\|_\tau$ *for every* $\mu > 0$;

(2) $\sum_{\lambda>0} e^{-\beta\lambda} \dim_\tau B_\lambda < 1$.

Then, for every $\beta > \beta_0$, all σ-KMS$_\beta$-states on A are of finite type, so they are in a one-to-one correspondence with the positive traces τ on A_c such that $\sum_{\lambda\in\Gamma_+} e^{-\beta\lambda} \dim_\tau X_\lambda = 1$.

Example: LCM monoids

Consider a left cancellative LCM monoid S, which means that for all $s, t \in S$ we have either $sS \cap tS = \emptyset$ or $sS \cap tS = rS$ for some $r \in S$.

Examples of such monoids include free and free abelian monoids, $ax + b$ semigroups over the rings of integers in number fields of class number 1, Artin–Tits monoids.

The semigroup C*-algebra $C^*(S)$ is generated by the isometries v_s, $s \in S$, satisfying the relations

$$v_e = 1, \quad v_s v_t = v_{st}, \quad \text{and} \quad v_s v_s^* v_t v_t^* = \begin{cases} v_r v_r^*, & \text{if } sS \cap tS = rS, \\ 0, & \text{if } sS \cap tS = \emptyset. \end{cases}$$

The elements $v_s v_t^*$, for all $s, t \in S$, span a dense subspace of $C^*(S)$.

Consider the dynamics σ defined by a homomorphism $N \colon S \to ([1, +\infty), \cdot)$,

$$\sigma_t^N(v_s) = N(s)^{it} v_s.$$

The group $\Gamma \subset \mathbb{R}$ is generated by $\log N(S)$, $\Gamma_+ = \log N(S)$.

We take B_λ to be equal to 0, if $\lambda \notin \log N(S) = \Gamma_+$, and to be the closed linear span of $v_s A_0$ with $\log N(s) = \lambda$ otherwise. Define

$$s \sim_N t \quad \text{if and only if} \quad sa = tb \quad \text{for some} \quad a, b \in \ker N.$$

Then representatives of the equivalence classes in $N^{-1}(e^\lambda)$ generate the same right A_0-module as the entire set $N^{-1}(e^\lambda)$ modulo vectors of semi-norm $\|\cdot\|_\tau$ zero. Hence

$$\dim_\tau B_\lambda \le |N^{-1}(e^\lambda)/\sim_N|.$$

The conclusion is that if $\beta > 0$ is such that

$$\zeta_N(\beta) - 1 = \sum_{s \in S/\sim_N} N(s)^{-\beta} - 1 = \sum_{s \in (S \setminus \ker N)/\sim_N} N(s)^{-\beta} < 1,$$

then we have an affine one-to-one correspondence between the KMS$_\beta$-states and the tracial states on $A_c = C^*(\ker N)$. This is actually true for all β such that $\zeta_N(\beta) < \infty$ (Stammeier–N).

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Crystals of groupoid C*-algebras

Assume $\mathcal{G}$ is a locally compact étale groupoid and we are given a 1-cocycle $\omega\colon \mathcal{G} \to \mathbb{R}_d$, where $\mathbb{R}_d$ is the group $\mathbb{R}$ with discrete topology. Define

$$\sigma_t(f)(g) = e^{it\omega(g)} f(g) \quad \text{for} \quad f \in C_c(\mathcal{G}) \quad \text{and} \quad g \in \mathcal{G}.$$

The **boundary set** (Renault, Laca–Larsen–N) of ω is defined by

$$Z = \{x \in \mathcal{G}^{(0)} \mid \omega \geq 0 \text{ on } \mathcal{G}_x\} = \{x \in \mathcal{G}^{(0)} \mid \omega \leq 0 \text{ on } \mathcal{G}^x\}.$$

Theorem

Assume $\mathcal{G}$ is a locally compact, not necessarily Hausdorff, étale groupoid and $\omega\colon \mathcal{G} \to \mathbb{R}_d$ is a 1-cocycle with boundary set Z. Consider the C^*-algebra $A = C^*(\mathcal{G})$ and the dynamics defined by ω. Then $A_c \cong C^*(\mathcal{G}_Z)$; in particular, $A_c \neq 0$ if and only if $Z \neq \emptyset$.

K-theory dreams

Dream No.1: The algebra $\Lambda(A) \subset \mathcal{L}(F)$ looks like a Toeplitz type algebra with the coefficient algebra A_c. Can we construct a class in $KK(A_c, A)$ similar to the class of the embedding $\mathbb{C} \hookrightarrow \mathcal{T}$?

Dream No.2: A_c can be thought of as obtained by freezing A. Does this give an asymptotic morphism $A \to A_c$?

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Functoriality problems: Instead of $\mathcal{T}$ we can equally well consider the compact operators $\mathcal{K} \subset \mathcal{T}$. The embedding $i\colon \mathcal{K} \to \mathcal{T}$ induces an isomorphism

$$i_c\colon \mathcal{K}_c = \mathbb{C} \to \mathcal{T}_c = \mathbb{C}.$$

But $KK(i) = 0$.

The same problem appears even in finite dimensions: For a finite dimensional Hilbert space H, a codimension one subspace H' and the embedding map $i\colon \mathrm{CAR}(H') \to \mathrm{CAR}(H)$, i induces an isomorphism i_c of the corresponding crystals, while the map

$$i_*\colon \mathbb{Z} \cong K_0(\mathrm{CAR}(H')) \to \mathbb{Z} \cong K_0(\mathrm{CAR}(H))$$

is the multiplication by ± 2.

Note also that if H is infinite dimensional, then $\mathrm{CAR}(H)_c = \mathbb{C}$, but $K_0(\mathrm{CAR}(H)) \cong \mathbb{Z}[1/2]$.

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More problems with functoriality

Consider a full D-D-C*-correspondence Y for a C*-algebra D, the Toeplitz–Pimsner algebra $A = \mathcal{T}_Y$ and the gauge action

$$\sigma_t(T_\xi) = e^{it} T_\xi, \quad \xi \in Y.$$

Then $A_c = D$, $\mathcal{F}$ is the standard Fock module $\bigoplus_{n\ge0} Y^{\otimes n}$, and the embedding map $D \to \mathcal{T}_Y$ is a KK-equivalence (Pimsner).

In particular, from the pair $(\mathcal{T}_Y, \sigma)$ we can reconstruct D and Y as a right C*-Hilbert D-module. But can we reconstruct the left action of D?

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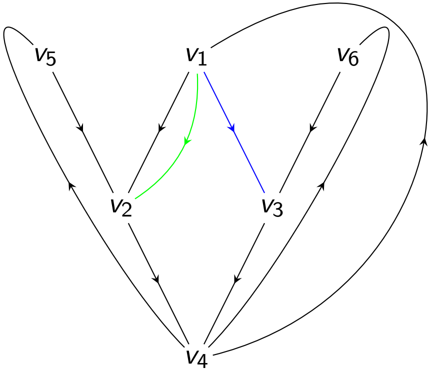
No! Brownlowe–Laca–Robertson–Sims:
Consider two graphs G and B that differ only in two edges (green and blue):



Consider the corresponding Toeplitz graphs C*-algebras A_B and A_G:
$D = C(\mathrm{vertices})$, $Y = C(\mathrm{edges})$.

Then there is an equivariant isomorphism $A_G \cong A_B$ that is the identity map on the crystals, but at the level of K_0 it gives the map $\mathbb{Z}^6 \to \mathbb{Z}^6$

$$\begin{pmatrix} 1 & 1 & -1 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

Circle actions

Assume now that we have a circle action σ. We will make two additional assumptions to simplify matters:

$$A = C^*(A_0, A_1) \qquad I_{-1} = \overline{A_1^* A_1} = A_0.$$

This implies that $I_1 \supset I_2 \supset \ldots$, hence

$$I = I_1 \quad \text{and} \quad A_c = A_0 / I_1.$$

We can view A_{-1} as an even Kasparov A_0-A_0-module with zero odd part. The class $[A_{-1}] \in KK(A_0, A_0)$ defines an endomorphism of $K_*(A_0) = KK_*(\mathbb{C}, A_0)$,

$$x \mapsto x \otimes_{A_0} [A_{-1}],$$

which we denote by β. This way $K_*(A_0)$ can be viewed as a $\mathbb{Z}[t]$-module, with t acting by β.

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Theorem

Assume that the following conditions are satisfied:

(1) *the* $\mathbb{Q}[t]$*-module* $K_*(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is finitely generated;*

(2) *the action of* t *on* $K_*(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ *has zero kernel and no nonzero fixed points.*

Then $K_*(A) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K_*(A_c) \otimes_{\mathbb{Z}} \mathbb{Q}$. *If* $K_*(A_0)$ *is a free* $\mathbb{Z}[t]$*-module, then* $K_*(A) \cong K_*(A_c)$.

How (un)reasonable are these assumptions?

The action of t on $K_*(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ has zero kernel if and only if the map $i_*\colon K_*(I_1) \to K_*(A_0)$ induced by the embedding $i\colon I_1 \to A_0$ is rationally injective. This is, for example, the case if A is unital and A_1 is finitely generated as a right A_0-module.

We also have

$$\bigcap_{n=0}^{\infty} t^n(K_*(A) \otimes_{\mathbb{Z}} \mathbb{Q}) = \bigcap_{n=0}^{\infty} \iota_{n*}(K_*(I_n)) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $i_n\colon I_n \to A_0$ is the inclusion map.

Lemma

The $*$-homomorphism $\Lambda\colon A \to \mathcal{L}(\mathcal{F})$ is faithful if and only if $\bigcap_{n=1}^{\infty} I_n = 0$.

The action of t on $K_*(A_0)\otimes_{\mathbb{Z}}\mathbb{Q}$ has zero kernel if and only if the map $i_*\colon K_*(I_1)\to K_*(A_0)$ induced by the embedding $i\colon I_1\to A_0$ is rationally injective. This is, for example, the case if A is unital and A_1 is finitely generated as a right A_0-module.

We also have

$$\bigcap_{n=0}^{\infty} t^n(K_*(A)\otimes_{\mathbb{Z}}\mathbb{Q}) = \bigcap_{n=0}^{\infty} \iota_{n*}(K_*(I_n))\otimes_{\mathbb{Z}}\mathbb{Q},$$

where $i_n\colon I_n\to A_0$ is the inclusion map.

Lemma

The $*$-homomorphism $\Lambda\colon A\to\mathcal{L}(\mathcal{F})$ is faithful if and only if $\bigcap_{n=1}^\infty I_n=0$.

The proof of the theorem follows from two exact sequences:

$$
\begin{array}{ccccc}
K_0(A_0) & \xrightarrow{\mathrm{id}-\beta} & K_0(A_0) & \longrightarrow & K_0(A) \\
\uparrow & & & & \downarrow \\
K_1(A) & \longleftarrow & K_1(A_0) & \xleftarrow{\mathrm{id}-\beta} & K_1(A_0)
\end{array}
$$

This is similar to Pimsner's 6-term exact sequence for the Cuntz–Pimsner algebras. (Alternatively, if A is separable, we can stabilize it and write as Exel's covariance algebra of a partial automorphism $\theta\colon A_0 \to I_1$ of A_0.)

$$
\begin{array}{ccccc}
K_0(A_0) & \xrightarrow{\beta} & K_0(A_0) & \longrightarrow & K_0(A_c) \\
\uparrow & & & & \downarrow \\
K_1(A_c) & \longleftarrow & K_1(A_0) & \xleftarrow{\beta} & K_1(A_0)
\end{array}
$$