A Galois group on meromorphic germs and locality evaluators

Sylvie Paycha
ongoing joint work with Li Guo and Bin Zhang

Noncommutative Geometry Seminar, February 9th 2021

Speer's analytic renormalisation [JMP 1967] revisited

Eugene Speer considers the perturbation-series expansion of the S matrix in a Lagrangian field theory (with non zero mass).

Speer's analytic renormalisation [JMP 1967] revisited

Eugene Speer considers the perturbation-series expansion of the S matrix in a Lagrangian field theory (with non zero mass).

Excerpt of Speer's article

In this paper we apply a method of defining divergent quantities which was originated by Riesz and has been used in various contexts by many authors. [....] We find it necessary to consider functions of several complex variables $z_1, \cdots, z_k$, one associated with each line of the Feynman graph. The main difficulty is the extension of the above [Riesz's] treatment of poles to the more complicated singularities which occur in several complex variables...

Speer's analytic renormalisation [JMP 1967] revisited

Eugene Speer considers the perturbation-series expansion of the S matrix in a Lagrangian field theory (with non zero mass).

Excerpt of Speer's article

In this paper we apply a method of defining divergent quantities which was originated by Riesz and has been used in various contexts by many authors. [....] We find it necessary to consider functions of several complex variables $z_1, \cdots, z_k$, one associated with each line of the Feynman graph. The main difficulty is the extension of the above [Riesz's] treatment of poles to the more complicated singularities which occur in several complex variables...

Speer shows [Theorem 1] that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty} \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ consisting of

Speer's analytic renormalisation [JMP 1967] revisited

Eugene Speer considers the perturbation-series expansion of the S matrix in a Lagrangian field theory (with non zero mass).

Excerpt of Speer's article

In this paper we apply a method of defining divergent quantities which was originated by Riesz and has been used in various contexts by many authors. [....] We find it necessary to consider functions of several complex variables $z_1, \cdots, z_k$, one associated with each line of the Feynman graph. The main difficulty is the extension of the above [Riesz's] treatment of poles to the more complicated singularities which occur in several complex variables...

Speer shows [Theorem 1] that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty} \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ consisting of

$$f : \mathbb{C}^k \to \mathbb{C},\ \exists\, s_I \in \mathbb{Z}_{\geq 0}, I \subset \{1, \cdots, k\}, \quad f(z_1, \cdots, z_k) \cdot \prod_{I \subset \{1, \cdots, k\}} \overbrace{\left(\sum_{i \in I} z_i\right)^{s_I}}^{{L_I}^{s_I}} \text{ holom. at zero}$$

E. Speer defines generalised evaluators $\mathcal{E}_k : \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k) \to \mathbb{C},\ k \in \mathbb{N}$ **linear forms compatible with the filtration**

Speer's analytic renormalisation [JMP 1967] revisited

Eugene Speer considers the perturbation-series expansion of the S matrix in a Lagrangian field theory (with non zero mass).

Excerpt of Speer's article

In this paper we apply a method of defining divergent quantities which was originated by Riesz and has been used in various contexts by many authors. [....] We find it necessary to consider functions of several complex variables $z_1, \cdots, z_k$, one associated with each line of the Feynman graph. The main difficulty is the extension of the above [Riesz's] treatment of poles to the more complicated singularities which occur in several complex variables...

Speer shows [Theorem 1] that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\rm Feyn}(\mathbb{C}^\infty) := \cup_{k=1}^\infty \mathcal{M}^{\rm Feyn}(\mathbb{C}^k)$ consisting of

$$f : \mathbb{C}^k \to \mathbb{C},\ \exists\, s_I \in \mathbb{Z}_{\geq 0}, I \subset \{1, \cdots, k\}, \quad f(z_1, \cdots, z_k) \cdot \prod_{I \subset \{1, \cdots, k\}} \overbrace{\left(\sum_{i \in I} z_i\right)^{s_I}}^{{L_I}^{s_I}} \text{ holom. at zero}$$

E. Speer defines **generalised evaluators** $\mathcal{E}_k : \mathcal{M}^{\rm Feyn}(\mathbb{C}^k) \to \mathbb{C}$, $k \in \mathbb{N}$ **linear forms compatible with the filtration**

1. **(extends ev_0)** $\mathcal{E}$ is the ordinary evaluation ev_0 at zero on holom. germs;

Speer's analytic renormalisation [JMP 1967] revisited

Eugene Speer considers the perturbation-series expansion of the S matrix in a Lagrangian field theory (with non zero mass).

Excerpt of Speer's article

In this paper we apply a method of defining divergent quantities which was originated by Riesz and has been used in various contexts by many authors. [....] We find it necessary to consider functions of several complex variables $z_1, \cdots, z_k$, one associated with each line of the Feynman graph. The main difficulty is the extension of the above [Riesz's] treatment of poles to the more complicated singularities which occur in several complex variables...

Speer shows [Theorem 1] that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\rm Feyn}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty} \mathcal{M}^{\rm Feyn}(\mathbb{C}^k)$ consisting of

$$f : \mathbb{C}^k \to \mathbb{C},\ \exists\, s_I \in \mathbb{Z}_{\geq 0}, I \subset \{1, \cdots, k\}, \quad f(z_1, \cdots, z_k) \cdot \prod_{I \subset \{1,\cdots,k\}} \overbrace{\left(\sum_{i \in I} z_i\right)^{s_I}}^{L_I^{s_I}} \text{ holom. at zero}$$

E. Speer defines **generalised evaluators** $\mathcal{E}_k : \mathcal{M}^{\rm Feyn}(\mathbb{C}^k) \to \mathbb{C}$, $k \in \mathbb{N}$ **linear forms compatible with the filtration**

1. **(extends ev_0)** $\mathcal{E}$ is the ordinary evaluation ev_0 at zero on holom. germs;
2. **(partial multiplicativity)** $\mathcal{E}(f_1 \cdot f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2)$ if f_1 and f_2 depend on different sets (later called independent) of variables z_i;

Speer's analytic renormalisation [JMP 1967] revisited

Eugene Speer considers the perturbation-series expansion of the S matrix in a Lagrangian field theory (with non zero mass).

Excerpt of Speer's article

In this paper we apply a method of defining divergent quantities which was originated by Riesz and has been used in various contexts by many authors. [....] We find it necessary to consider functions of several complex variables $z_1, \cdots, z_k$, one associated with each line of the Feynman graph. The main difficulty is the extension of the above [Riesz's] treatment of poles to the more complicated singularities which occur in several complex variables...

Speer shows [Theorem 1] that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty} \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ consisting of

$$f : \mathbb{C}^k \to \mathbb{C}, \exists\, s_I \in \mathbb{Z}_{\geq 0}, I \subset \{1, \cdots, k\}, \quad f(z_1, \cdots, z_k) \cdot \prod_{I \subset \{1,\cdots,k\}} \overbrace{\left(\sum_{i \in I} z_i\right)^{s_I}}^{L_I{}^{s_I}} \text{ holom. at zero}$$

E. Speer defines generalised evaluators $\mathcal{E}_k : \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k) \to \mathbb{C}$, $k \in \mathbb{N}$ **linear forms compatible with the filtration**

1. **(extends ev_0)** $\mathcal{E}$ is the ordinary evaluation ev_0 at zero on holom. germs;
2. **(partial multiplicativity)** $\mathcal{E}(f_1 \cdot f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2)$ if f_1 and f_2 depend on different sets (later called independent) of variables z_i;
3. $\mathcal{E}$ is invariant under permutations of the variables $\mathcal{E}_k \circ \sigma^* = \mathcal{E}_k$ for any $\sigma \in \Sigma_k$, with $\sigma^* f(z_1, \cdots, z_k) := f(z_{\sigma(1)}, \cdots, z_{\sigma(k)})$;

Speer's analytic renormalisation [JMP 1967] revisited

Eugene Speer considers the perturbation-series expansion of the S matrix in a Lagrangian field theory (with non zero mass).

Excerpt of Speer's article

In this paper we apply a method of defining divergent quantities which was originated by Riesz and has been used in various contexts by many authors. [....] We find it necessary to consider functions of several complex variables $z_1, \cdots, z_k$, one associated with each line of the Feynman graph. The main difficulty is the extension of the above [Riesz's] treatment of poles to the more complicated singularities which occur in several complex variables...

Speer shows [Theorem 1] that the divergent expressions lie in the filtered algebra $\mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^\infty) := \cup_{k=1}^{\infty} \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k)$ consisting of

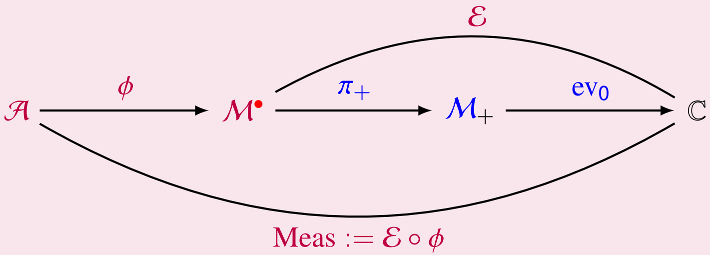
$$f : \mathbb{C}^k \to \mathbb{C},\ \exists\, s_I \in \mathbb{Z}_{\geq 0}, I \subset \{1, \cdots, k\}, \quad f(z_1, \cdots, z_k) \cdot \prod_{I \subset \{1, \cdots, k\}} \overbrace{\left(\sum_{i \in I} z_i \right)^{s_I}}^{{L_I}^{s_I}} \text{ holom. at zero}$$

E. Speer defines **generalised evaluators** $\mathcal{E}_k : \mathcal{M}^{\mathrm{Feyn}}(\mathbb{C}^k) \to \mathbb{C}$, $k \in \mathbb{N}$ **linear forms compatible with the filtration**

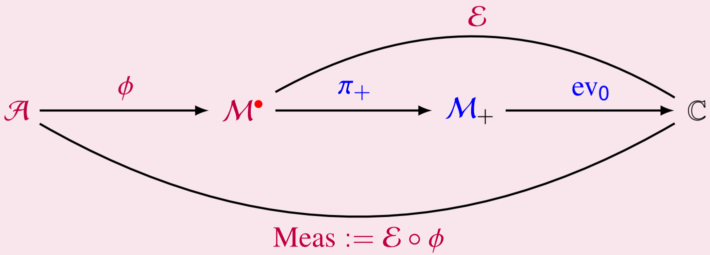
1. **(extends ev_0)** $\mathcal{E}$ is the ordinary evaluation ev_0 at zero on holom. germs;
2. **(partial multiplicativity)** $\mathcal{E}(f_1 \cdot f_2) = \mathcal{E}(f_1) \cdot \mathcal{E}(f_2)$ if f_1 and f_2 depend on different sets (later called independent) of variables z_i;
3. $\mathcal{E}$ is invariant under permutations of the variables $\mathcal{E}_k \circ \sigma^* = \mathcal{E}_k$ for any $\sigma \in \Sigma_k$, with $\sigma^* f(z_1, \cdots, z_k) := f(z_{\sigma(1)}, \cdots, z_{\sigma(k)})$;
4. (continuity) If $f_n(\vec{z}_k) \cdot \prod_{I \subset \{1, \cdots, k\}} \left(\sum_{i \in I} z_i\right)^{s_I} \underset{n \to \infty}{\overset{\text{uniformly}}{\longrightarrow}} g(\vec{z}_k)$ as holomorphic germs, then $\mathcal{E}_k(f_n) \underset{n \to \infty}{\longrightarrow} \mathcal{E}_k(\lim_{n \to \infty} f_n)$.

I. Framework and protagonists

The abstract setup

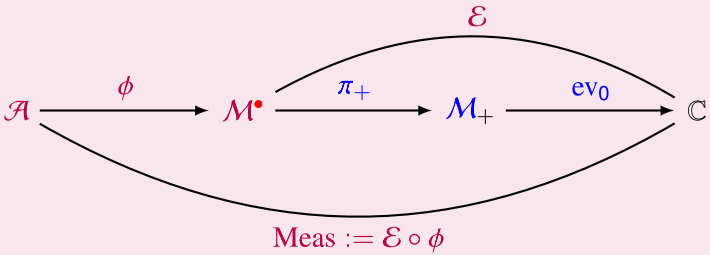


The abstract setup



The protagonists

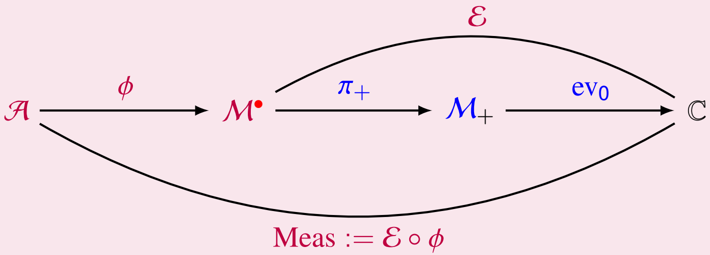
The abstract setup



The protagonists

- $(\mathcal{A}, \vee)$ is an algebra consisting of Feynman graphs, rooted trees or cones equipped with a concatenation $\vee$;

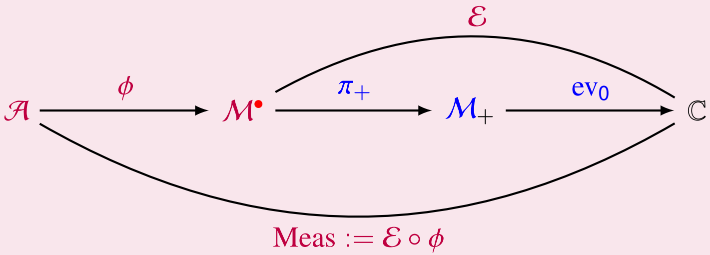
The abstract setup



The protagonists

- $(\mathcal{A}, \vee)$ is an algebra consisting of Feynman graphs, rooted trees or cones equipped with a concatenation $\vee$;
- $(\mathcal{M}^\bullet, \cdot)$ is an algebra of meromorphic germs at zero with pointwise product;

The abstract setup



The protagonists

- $(\mathcal{A}, \vee)$ is an algebra consisting of Feynman graphs, rooted trees or cones equipped with a concatenation $\vee$;
- $(\mathcal{M}^\bullet, \cdot)$ is an algebra of meromorphic germs at zero with pointwise product;
- $\phi : (\mathcal{A}, \vee) \longrightarrow (\mathcal{M}^\bullet, \cdot)$ is a morphism given by Feynman integrals, branched zeta functions or conical zeta functions.

Our main protagonists: Meromorphic germs

Our main protagonists: Meromorphic germs

Meromorphic germs in several variables with linear poles

Our main protagonists: Meromorphic germs

Meromorphic germs in several variables with linear poles

The target algebra $(\mathcal{M}^{\bullet}, \cdot)$ is an algebra of meromorphic germs at zero

Our main protagonists: Meromorphic germs

Meromorphic germs in several variables with linear poles

The target algebra $(\mathcal{M}^{\bullet}, \cdot)$ is an algebra of meromorphic germs at zero

- 1-variable meromorphic germs in **A. Connes** and **D. Kreimer**'s ABF approach (1998)

Our main protagonists: Meromorphic germs

Meromorphic germs in several variables with linear poles

The target algebra $(\mathcal{M}^\bullet, \cdot)$ is an algebra of meromorphic germs at zero

- 1-variable meromorphic germs in **A. Connes** and **D. Kreimer**'s ABF approach (1998)
- Here (and in Speer's analytic renorm) $\mathcal{M}^\bullet \subset \mathcal{M}(\mathbb{C}^\infty)$ is an algebra of meromorphic germs at zero in several variables.

Prescribed pole structure: linear poles

Our main protagonists: Meromorphic germs

Meromorphic germs in several variables with linear poles

The target algebra $(\mathcal{M}^\bullet, \cdot)$ is an algebra of meromorphic germs at zero

- 1-variable meromorphic germs in **A. Connes** and **D. Kreimer**'s ABF approach (1998)
- Here (and in Speer's analytic renorm) $\mathcal{M}^\bullet \subset \mathcal{M}(\mathbb{C}^\infty)$ is an algebra of meromorphic germs at zero in several variables.

Prescribed pole structure: linear poles

$$f \in \mathcal{M}(\mathbb{C}^k) \Longleftrightarrow f \cdot \left(\prod_{i=1}^{n} L_i^{s_i}\right) : \mathbb{C}^k \to \mathbb{C}\ \text{holom.},\quad L_i : \mathbb{C}^k \to \mathbb{C}, i = 1, \cdots, n\ \text{linear}$$

Our main protagonists: Meromorphic germs

Meromorphic germs in several variables with linear poles

The target algebra $(\mathcal{M}^\bullet, \cdot)$ is an algebra of meromorphic germs at zero

- 1-variable meromorphic germs in **A. Connes** and **D. Kreimer**'s ABF approach (1998)
- Here (and in Speer's analytic renorm) $\mathcal{M}^\bullet \subset \mathcal{M}(\mathbb{C}^\infty)$ is an algebra of meromorphic germs at zero in several variables.

Prescribed pole structure: linear poles

$$f \in \mathcal{M}(\mathbb{C}^k) \Longleftrightarrow f \cdot \left(\prod_{i=1}^{n} L_i^{s_i}\right) : \mathbb{C}^k \to \mathbb{C} \text{ holom.}, \quad L_i : \mathbb{C}^k \to \mathbb{C}, i = 1, \cdots, n \text{ linear}$$

- cones (Laplace transforms on polyhedral cones) [GPZ DJM 2017]: $L_i(\vec{z}) = \sum_{j=1}^{k} a_{j_i} z_{j_i} \rightsquigarrow \mathcal{M}$;

Our main protagonists: Meromorphic germs

Meromorphic germs in several variables with linear poles

The target algebra $(\mathcal{M}^\bullet, \cdot)$ is an algebra of meromorphic germs at zero

- 1-variable meromorphic germs in **A. Connes** and **D. Kreimer**'s ABF approach (1998)
- Here (and in Speer's analytic renorm) $\mathcal{M}^\bullet \subset \mathcal{M}(\mathbb{C}^\infty)$ is an algebra of meromorphic germs at zero in several variables.

Prescribed pole structure: linear poles

$$f \in \mathcal{M}(\mathbb{C}^k) \Longleftrightarrow f \cdot \left(\prod_{i=1}^{n} L_i^{s_i} \right) : \mathbb{C}^k \to \mathbb{C} \text{ holom.}, \ \ L_i : \mathbb{C}^k \to \mathbb{C}, i = 1, \cdots, n \text{ linear}$$

- cones (Laplace transforms on polyhedral cones) [GPZ DJM 2017]: $L_i(\vec{z}) = \sum_{j=1}^{k} a_{j_i} z_{j_i} \rightsquigarrow \mathcal{M}$;
- Feynman graphs (Feynman integrals) [Speer JMP 1967]:$L_i(\vec{z}) = \sum_{j_i \in J_i} z_{j_i}$ with $J_i \subset \{1, \cdots, k\} \rightsquigarrow \mathcal{M}^{\mathrm{Feyn}}$;

Our main protagonists: Meromorphic germs

Meromorphic germs in several variables with linear poles

The target algebra $(\mathcal{M}^\bullet, \cdot)$ is an algebra of meromorphic germs at zero

- 1-variable meromorphic germs in **A. Connes** and **D. Kreimer**'s ABF approach (1998)
- Here (and in Speer's analytic renorm) $\mathcal{M}^\bullet \subset \mathcal{M}(\mathbb{C}^\infty)$ is an algebra of meromorphic germs at zero in several variables.

Prescribed pole structure: linear poles

$$f \in \mathcal{M}(\mathbb{C}^k) \Longleftrightarrow f \cdot \left(\prod_{i=1}^{n} L_i^{s_i}\right) : \mathbb{C}^k \to \mathbb{C} \text{ holom.}, \quad L_i : \mathbb{C}^k \to \mathbb{C}, i = 1, \cdots, n \text{ linear}$$

- cones (Laplace transforms on polyhedral cones) [GPZ DJM 2017]: $L_i(\vec{z}) = \sum_{j=1}^{k} a_{j_i} z_{j_i} \rightsquigarrow \mathcal{M}$;
- Feynman graphs (Feynman integrals) [Speer JMP 1967]:$L_i(\vec{z}) = \sum_{j_i \in J_i} z_{j_i}$ with $J_i \subset \{1, \cdots, k\} \rightsquigarrow \mathcal{M}^{\mathrm{Feyn}}$;
- Chen integrals (Kreimer's toy model) $x_{j_1} \leq x_{j_1} + x_{j_2} \cdots \leq x_{j_1} + x_{j_2} + \cdots + x_{j_i}$ [CGPZ JMP 2020]: $L_i(\vec{z}) = \sum_{l=1}^{i} z_{j_l}; \rightsquigarrow \mathcal{M}^{\mathrm{Chen}}$;

Our main protagonists: Meromorphic germs

Meromorphic germs in several variables with linear poles

The target algebra $(\mathcal{M}^\bullet, \cdot)$ is an algebra of meromorphic germs at zero

- 1-variable meromorphic germs in **A. Connes** and **D. Kreimer**'s ABF approach (1998)
- Here (and in Speer's analytic renorm) $\mathcal{M}^\bullet \subset \mathcal{M}(\mathbb{C}^\infty)$ is an algebra of meromorphic germs at zero in several variables.

Prescribed pole structure: linear poles

$$f \in \mathcal{M}(\mathbb{C}^k) \Longleftrightarrow f \cdot \left(\prod_{i=1}^{n} L_i^{s_i}\right) : \mathbb{C}^k \to \mathbb{C} \text{ holom.}, \quad L_i : \mathbb{C}^k \to \mathbb{C}, i = 1, \cdots, n \text{ linear}$$

- cones (Laplace transforms on polyhedral cones) [GPZ DJM 2017]: $L_i(\vec{z}) = \sum_{j=1}^k a_{j_i} z_{j_i} \rightsquigarrow \mathcal{M}$;
- Feynman graphs (Feynman integrals) [Speer JMP 1967]:$L_i(\vec{z}) = \sum_{j_i \in J_i} z_{j_i}$ with $J_i \subset \{1, \cdots, k\} \rightsquigarrow \mathcal{M}^{\text{Feyn}}$;
- Chen integrals (Kreimer's toy model) $x_{j_1} \leq x_{j_1} + x_{j_2} \cdots \leq x_{j_1} + x_{j_2} + \cdots + x_{j_i}$ [CGPZ JMP 2020]: $L_i(\vec{z}) = \sum_{l=1}^i z_{j_l}; \rightsquigarrow \mathcal{M}^{\text{Chen}}$;
- branched zeta functions (higher zeta functions modelled on rooted trees) [Irma lectures EMS 2020] : $L_i(\vec{z}) = \sum_{v \leq v_i} z_v$, v vertex of the tree.

II. A guiding principle: locality

Locality and independence

We want to evaluate meromorphic germs at poles according to the principle of locality which roughly says that an object is directly influenced only by its immediate surroundings:

Locality and independence

We want to evaluate meromorphic germs at poles according to the principle of locality which roughly says that an object is directly influenced only by its immediate surroundings:

- "*two events separated in space can be measured independently*"

Locality and independence

We want to evaluate meromorphic germs at poles according to the principle of locality which roughly says that an object is directly influenced only by its immediate surroundings:

- "*two events separated in space can be measured independently*"

reinterpreted here as follows:

Principle of locality: factorisation on independent events

$$\underbrace{a \text{ and } b}_{\in \mathcal{A}} \textit{ independent} \underset{\text{factorisation}}{\Longrightarrow} \mathrm{Meas}\ \underbrace{(a \vee b)}_{\text{concatenation}} = \mathrm{Meas}(a) \cdot \mathrm{Meas}(b).$$

Locality and independence

We want to evaluate meromorphic germs at poles according to the principle of locality which roughly says that an object is directly influenced only by its immediate surroundings:

- *"two events separated in space can be measured independently"*

reinterpreted here as follows:

Principle of locality: factorisation on independent events

$$\underbrace{a \text{ and } b}_{\in \mathcal{A}} \textit{ independent} \underset{\text{factorisation}}{\Longrightarrow} \text{Meas}\,\underbrace{(a \vee b)}_{\text{concatenation}} = \text{Meas}(a) \cdot \text{Meas}(b).$$

To generalise this statement to an abstract setting, we need two ingredients:

- **locality:** independence of events $\rightsquigarrow a \top b$ for some locality (binary symmetric) relation $\top$;

Locality and independence

We want to evaluate meromorphic germs at poles according to the principle of locality which roughly says that an object is directly influenced only by its immediate surroundings:

- "*two events separated in space can be measured independently*"

reinterpreted here as follows:

Principle of locality: factorisation on independent events

$$\underbrace{a \text{ and } b}_{\in \mathcal{A}} \textit{ independent} \underset{\text{factorisation}}{\Longrightarrow} \mathrm{Meas}\,\underbrace{(a \vee b)}_{\text{concatenation}} = \mathrm{Meas}(a) \cdot \mathrm{Meas}(b).$$

To generalise this statement to an abstract setting, we need two ingredients:

- **locality:** independence of events $\rightsquigarrow a \top b$ for some locality (binary symmetric) relation $\top$;
- **evaluator:** measurement $\rightsquigarrow$ locality evaluator $\mathcal{E}$ on meromorphic germs.

Locality and independence

We want to evaluate meromorphic germs at poles according to the principle of locality which roughly says that an object is directly influenced only by its immediate surroundings:

- "*two events separated in space can be measured independently*"

reinterpreted here as follows:

Principle of locality: factorisation on independent events

$$\underbrace{a \text{ and } b}_{\in\mathcal{A}} \textit{ independent } \underset{\text{factorisation}}{\Longrightarrow} \text{Meas}\underbrace{(a \vee b)}_{\text{concatenation}} = \text{Meas}(a) \cdot \text{Meas}(b).$$

To generalise this statement to an abstract setting, we need two ingredients:

- **locality:** independence of events $\rightsquigarrow a \top b$ for some locality (binary symmetric) relation $\top$;
- **evaluator:** measurement $\rightsquigarrow$ locality evaluator $\mathcal{E}$ on meromorphic germs.

Principle of locality revisited: locality evaluators

$f \top g \Longrightarrow \mathcal{E}(f \cdot g) = \mathcal{E}(f)\,\mathcal{E}(g)$ for two meromorphic germs f and g in $\mathcal{M}^\bullet$.

Locality and independence

We want to evaluate meromorphic germs at poles according to the principle of locality which roughly says that an object is directly influenced only by its immediate surroundings:

- "*two events separated in space can be measured independently*"

reinterpreted here as follows:

Principle of locality: factorisation on independent events

$$\underbrace{a \text{ and } b}_{\in \mathcal{A}} \text{ independent} \underset{\text{factorisation}}{\Longrightarrow} \text{Meas}\underbrace{(a \vee b)}_{\text{concatenation}} = \text{Meas}(a) \cdot \text{Meas}(b).$$

To generalise this statement to an abstract setting, we need two ingredients:

- **locality:** independence of events $\rightsquigarrow a \top b$ for some locality (binary symmetric) relation $\top$;
- **evaluator:** measurement $\rightsquigarrow$ locality evaluator $\mathcal{E}$ on meromorphic germs.

Principle of locality revisited: locality evaluators

$f \top g \Longrightarrow \mathcal{E}(f \cdot g) = \mathcal{E}(f)\,\mathcal{E}(g)$ for two meromorphic germs f and g in $\mathcal{M}^\bullet$.

Claim: On certain algebras of meromorphic germs with a prescriped type of pole at zero, modulo a Galois transformation, any $\perp$-locality evaluator at the poles is determined by a minimal subtraction scheme.

Locality on/independence of meromorphic germs

Locality on/independence of meromorphic germs

Meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,

Locality on/independence of meromorphic germs

Meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1,\cdots,\ell_n)}{L_1^{s_1}\cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms with real coefficients (lie in $\mathcal{L}(\mathbb{C}^k)$).

Locality on/independence of meromorphic germs

Meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms with real coefficients (lie in $\mathcal{L}(\mathbb{C}^k)$).
- **Example:** $(z_1, z_2) \longmapsto \frac{z_1 - z_2}{z_1 + z_2}$.

Locality on/independence of meromorphic germs

Meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1,\cdots,\ell_n)}{L_1^{s_1}\cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms with real coefficients (lie in $\mathcal{L}(\mathbb{C}^k)$).
- **Example:** $(z_1, z_2) \longmapsto \frac{z_1 - z_2}{z_1 + z_2}$.

Locality on meromorphic germs: orthogonality

- **Dependence** set $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.

Locality on/independence of meromorphic germs

Meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1,\cdots,\ell_n)}{L_1^{s_1}\cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms with real coefficients (lie in $\mathcal{L}(\mathbb{C}^k)$).
- **Example:** $(z_1, z_2) \longmapsto \frac{z_1 - z_2}{z_1 + z_2}$.

Locality on meromorphic germs: orthogonality

- **Dependence** set $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.
- Q inner product on $\mathbb{R}^k$ induces one on $\mathcal{L}(\mathbb{C}^k)$ and we set $f_1 \perp^Q f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp^Q \mathrm{Dep}(f_2)$.

Locality on/independence of meromorphic germs

Meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms with real coefficients (lie in $\mathcal{L}(\mathbb{C}^k)$).
- **Example:** $(z_1, z_2) \longmapsto \frac{z_1 - z_2}{z_1 + z_2}$.

Locality on meromorphic germs: orthogonality

- **Dependence** set $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.
- Q inner product on $\mathbb{R}^k$ induces one on $\mathcal{L}(\mathbb{C}^k)$ and we set $f_1 \perp^Q f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp^Q \mathrm{Dep}(f_2)$.

Take Q the canonical inner product on $\mathbb{R}^k$. Then $(z_1 - z_2) \perp^Q (z_1 + z_2)$.

Locality on/independence of meromorphic germs

Meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1,\cdots,\ell_n)}{L_1^{s_1}\cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms with real coefficients (lie in $\mathcal{L}(\mathbb{C}^k)$).
- **Example:** $(z_1, z_2) \longmapsto \frac{z_1 - z_2}{z_1 + z_2}$.

Locality on meromorphic germs: orthogonality

- **Dependence** set $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.
- Q inner product on $\mathbb{R}^k$ induces one on $\mathcal{L}(\mathbb{C}^k)$ and we set $f_1 \perp^Q f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp^Q \mathrm{Dep}(f_2)$.

Take Q the canonical inner product on $\mathbb{R}^k$. Then $(z_1 - z_2) \perp^Q (z_1 + z_2)$.

Speer's locality: separation of variables

$(z_1 - z_2) \perp^{\mathrm{Speer}} (z_3 + z_4) \Rightarrow (z_1 - z_2) \perp^Q (z_3 + z_4)$.

III. Statement and ingredients for its proof

Statement of the main result

Data

- $\left(\mathcal{M}^\bullet, \perp^Q\right)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen, Feynman);

Statement of the main result

Data

- $\left(\mathcal{M}^\bullet, \perp^Q\right)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen, Feynman);
- $\mathcal{M}_+ \subset \mathcal{M}(\mathbb{C}^\infty)$ the algebra of holomorphic germs at zero;

Statement of the main result

Data

- $\left(\mathcal{M}^{\bullet}, \perp^{Q}\right)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen, Feynman);
- $\mathcal{M}_{+} \subset \mathcal{M}(\mathbb{C}^{\infty})$ the algebra of holomorphic germs at zero;
- the evaluation at zero: $\mathrm{ev}_0 : \mathcal{M}_{+} \to \mathbb{C}$;

Statement of the main result

Data

- $\left(\mathcal{M}^\bullet, \perp^Q\right)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen, Feynman);
- $\mathcal{M}_+ \subset \mathcal{M}(\mathbb{C}^\infty)$ the algebra of holomorphic germs at zero;
- the evaluation at zero: $\mathrm{ev}_0 : \mathcal{M}_+ \to \mathbb{C}$;
- $\perp^Q$ induces a splitting $\mathcal{M}^\bullet = \mathcal{M}_+ \oplus \mathcal{M}_-^Q$, where $\mathcal{M}_-^Q$ is the set of polar germs;

Statement of the main result

Data

- $\left(\mathcal{M}^\bullet, \perp^Q\right)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen, Feynman);
- $\mathcal{M}_+ \subset \mathcal{M}(\mathbb{C}^\infty)$ the algebra of holomorphic germs at zero;
- the evaluation at zero: $\mathrm{ev}_0 : \mathcal{M}_+ \to \mathbb{C}$;
- $\perp^Q$ induces a splitting $\mathcal{M}^\bullet = \mathcal{M}_+ \oplus \mathcal{M}_-^Q$, where $\mathcal{M}_-^Q$ is the set of polar germs; $\pi_+ : \mathcal{M}^\bullet \longrightarrow \mathcal{M}_+$ the induced projection;

Statement of the main result

Data

- $\left(\mathcal{M}^\bullet, \perp^Q\right)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen, Feynman);
- $\mathcal{M}_+ \subset \mathcal{M}(\mathbb{C}^\infty)$ the algebra of holomorphic germs at zero;
- the evaluation at zero: $\mathrm{ev}_0 : \mathcal{M}_+ \to \mathbb{C}$;
- $\perp^Q$ induces a splitting $\mathcal{M}^\bullet = \mathcal{M}_+ \oplus \mathcal{M}_-^Q$, where $\mathcal{M}_-^Q$ is the set of polar germs; $\pi_+ : \mathcal{M}^\bullet \longrightarrow \mathcal{M}_+$ the induced projection;
- the Galois group $\mathrm{Gal}^{\perp^Q}(\mathcal{M}^\bullet/\mathcal{M}_+)$ of (locality) transformations of $\left(\mathcal{M}^\bullet, \perp^Q\right)$.

Statement of the main result

Data

- $\left(\mathcal{M}^\bullet, \perp^Q\right)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen, Feynman);
- $\mathcal{M}_+ \subset \mathcal{M}(\mathbb{C}^\infty)$ the algebra of holomorphic germs at zero;
- the evaluation at zero: $\mathrm{ev}_0 : \mathcal{M}_+ \to \mathbb{C}$;
- $\perp^Q$ induces a splitting $\mathcal{M}^\bullet = \mathcal{M}_+ \oplus \mathcal{M}_-^Q$, where $\mathcal{M}_-^Q$ is the set of polar germs; $\pi_+ : \mathcal{M}^\bullet \longrightarrow \mathcal{M}_+$ the induced projection;
- the Galois group $\mathrm{Gal}^{\perp^Q}(\mathcal{M}^\bullet/\mathcal{M}_+)$ of (locality) transformations of $\left(\mathcal{M}^\bullet, \perp^Q\right)$.

Theorem [Guo, S.P., Zhang 2021]

When Q is the canonical inner product, a locality evaluator at zero $\mathcal{E} : \mathcal{M}^\bullet \longrightarrow \mathbb{C}$ i.e. a linear form which i) extends the ordinary evaluation ev_0 at zero and ii) factorises on independent germs is of the form:

Statement of the main result

Data

- $\left(\mathcal{M}^\bullet, \perp^Q\right)$ an (locality) algebra of meromorphic germs at zero with a prescribed type of poles (e.g. Chen, Feynman);
- $\mathcal{M}_+ \subset \mathcal{M}(\mathbb{C}^\infty)$ the algebra of holomorphic germs at zero;
- the evaluation at zero: $\mathrm{ev}_0 : \mathcal{M}_+ \to \mathbb{C}$;
- $\perp^Q$ induces a splitting $\mathcal{M}^\bullet = \mathcal{M}_+ \oplus \mathcal{M}_-^Q$, where $\mathcal{M}_-^Q$ is the set of polar germs; $\pi_+ : \mathcal{M}^\bullet \longrightarrow \mathcal{M}_+$ the induced projection;
- the Galois group $\mathrm{Gal}^{\perp^Q}(\mathcal{M}^\bullet/\mathcal{M}_+)$ of (locality) transformations of $\left(\mathcal{M}^\bullet, \perp^Q\right)$.

Theorem [Guo, S.P., Zhang 2021]

When Q is the canonical inner product, a locality evaluator at zero $\mathcal{E} : \mathcal{M}^\bullet \longrightarrow \mathbb{C}$ i.e. a linear form which i) extends the ordinary evaluation ev_0 at zero and ii) factorises on independent germs is of the form: $\mathcal{E} = \underbrace{\mathrm{ev}_0 \circ \pi_+}_{\text{minimal subtraction}} \circ \underbrace{T_{\mathcal{E}}}_{\mathrm{Gal}^\perp(\mathcal{M}^\bullet/\mathcal{M}_+)}$.

First ingredient for the proof: locality algebras

First ingredient for the proof: locality algebras

A locality set is a pair $(X, \top \subset X \times X)$ where $\top$ is a binary symmetric relation on X. **Notation:** $x^\top := \{y \in X,\ y \top x\}$.

First ingredient for the proof: locality algebras

A locality set is a pair $(X, \top \subset X \times X)$ where $\top$ is a binary symmetric relation on X. **Notation:** $x^{\top} := \{y \in X, \; y \top x\}$.

Ex: separating locality relations [Clavier, Guo, S.P., Zhang (2020)]

Disjointness on the power set $X := \mathcal{P}(S)$ of a set S; **Orthogonality** on the set $X := G(V)$ of subspaces of a euclidean space V.

First ingredient for the proof: locality algebras

A locality set is a pair $(X, \top \subset X \times X)$ where $\top$ is a binary symmetric relation on X. **Notation:** $x^{\top} := \{y \in X,\ y \top x\}$.

Ex: separating locality relations [Clavier, Guo, S.P., Zhang (2020)]

Disjointness on the power set $X := \mathcal{P}(S)$ of a set S; **Orthogonality** on the set $X := G(V)$ of subspaces of a euclidean space V.

Locality algebras and modules

Let $(R, m, +)$ be a ring. $(A, \top_A)$ is a locality R-algebra if

First ingredient for the proof: locality algebras

A locality set is a pair $(X, \top \subset X \times X)$ where $\top$ is a binary symmetric relation on X. **Notation:** $x^{\top} := \{y \in X,\ y \top x\}$.

Ex: separating locality relations [Clavier, Guo, S.P., Zhang (2020)]

Disjointness on the power set $X := \mathcal{P}(S)$ of a set S; **Orthogonality** on the set $X := G(V)$ of subspaces of a euclidean space V.

Locality algebras and modules

Let $(R, m, +)$ be a ring. $(A, \top_A)$ is a locality R-algebra if

- A is a locality vector space i.e., a vector space such that $S^{\top}$ is a vector space for any subset $S \subset A$;

First ingredient for the proof: locality algebras

A locality set is a pair $(X, \top \subset X \times X)$ where $\top$ is a binary symmetric relation on X. **Notation:** $x^{\top} := \{y \in X, \, y \top x\}$.

Ex: separating locality relations [Clavier, Guo, S.P., Zhang (2020)]

Disjointness on the power set $X := \mathcal{P}(S)$ of a set S; **Orthogonality** on the set $X := G(V)$ of subspaces of a euclidean space V.

Locality algebras and modules

Let $(R, m, +)$ be a ring. $(A, \top_A)$ is a locality R-algebra if

- A is a locality vector space i.e., a vector space such that $S^{\top}$ is a vector space for any subset $S \subset A$;
- the product $m : R \times R \to R$ extends to a partial product $m_A : \top_A \to A$ which is R-linear.

First ingredient for the proof: locality algebras

A locality set is a pair $(X, \top \subset X \times X)$ where $\top$ is a binary symmetric relation on X. **Notation:** $x^\top := \{y \in X,\ y \top x\}$.

Ex: separating locality relations [Clavier, Guo, S.P., Zhang (2020)]

Disjointness on the power set $X := \mathcal{P}(S)$ of a set S; **Orthogonality** on the set $X := G(V)$ of subspaces of a euclidean space V.

Locality algebras and modules

Let $(R, m, +)$ be a ring. $(A, \top_A)$ is a locality R-algebra if

- A is a locality vector space i.e., a vector space such that $S^\top$ is a vector space for any subset $S \subset A$;
- the product $m : R \times R \to R$ extends to a partial product $m_A : \top_A \to A$ which is R-linear.

A is generated by a subset $S \subset A$ as a $\top_A$-**locality** R **-module** if

First ingredient for the proof: locality algebras

A locality set is a pair $(X, \top \subset X \times X)$ where $\top$ is a binary symmetric relation on X. **Notation:** $x^\top := \{y \in X,\ y \top x\}$.

Ex: separating locality relations [Clavier, Guo, S.P., Zhang (2020)]

Disjointness on the power set $X := \mathcal{P}(S)$ of a set S; **Orthogonality** on the set $X := G(V)$ of subspaces of a euclidean space V.

Locality algebras and modules

Let $(R, m, +)$ be a ring. $(A, \top_A)$ is a locality R-algebra if

- A is a locality vector space i.e., a vector space such that $S^\top$ is a vector space for any subset $S \subset A$;
- the product $m : R \times R \to R$ extends to a partial product $m_A : \top_A \to A$ which is R-linear.

A is generated by a subset $S \subset A$ as a $\top_A$-**locality** R **-module** if

$$A = \langle S \rangle_R{}^{\top_A} := \sum_{s \in S} (s^{\top_A} \cap R) \cdot s.$$

Second ingredient for the proof: Laurent expansions

Second ingredient for the proof: Laurent expansions

Polar germs

A Q-**polar germ** in $\mathcal{M}(\mathbb{C}^k)$:

Second ingredient for the proof: Laurent expansions

Polar germs

A Q-**polar germ** in $\mathcal{M}(\mathbb{C}^k)$: $P := \frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}} = h \cdot S$, with

Second ingredient for the proof: Laurent expansions

Polar germs

A Q-**polar germ** in $\mathcal{M}(\mathbb{C}^k)$: $P := \frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}} = h \cdot S$, with $S := \frac{1}{L_1^{s_1}\cdots L_n^{s_n}} \in \mathcal{S}$, such that

- h is holomorphic at zero i.e. $h \in \mathcal{M}_+(\mathbb{C}^k)$;

Second ingredient for the proof: Laurent expansions

Polar germs

A Q-**polar germ** in $\mathcal{M}(\mathbb{C}^k)$: $P := \frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}} = h \cdot S$, with $S := \frac{1}{L_1^{s_1}\cdots L_n^{s_n}} \in \mathcal{S}$, such that

- h is holomorphic at zero i.e. $h \in \mathcal{M}_+(\mathbb{C}^k)$;
- $\ell_1, \cdots, \ell_m, L_1, \cdots, L_n$ are linearly independent and $\langle \ell_1, \cdots, \ell_m \rangle \perp^Q \langle L_1, \cdots, L_n \rangle$ (locality independence).

Second ingredient for the proof: Laurent expansions

Polar germs

A Q-**polar germ** in $\mathcal{M}(\mathbb{C}^k)$: $P := \frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}} = h \cdot S$, with $S := \frac{1}{L_1^{s_1}\cdots L_n^{s_n}} \in \mathcal{S}$, such that

- h is holomorphic at zero i.e. $h \in \mathcal{M}_+(\mathbb{C}^k)$;
- $\ell_1, \cdots, \ell_m$, $L_1, \cdots, L_n$ are linearly independent and $\langle \ell_1, \cdots, \ell_m \rangle \perp^Q \langle L_1, \cdots, L_n \rangle$ (locality independence).

They generate the subspace $\mathcal{M}_-^Q(\mathbb{C}^k) \subset \mathcal{M}(\mathbb{C}^k)$.

Second ingredient for the proof: Laurent expansions

Polar germs

A Q-**polar germ** in $\mathcal{M}(\mathbb{C}^k)$: $P := \frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}} = h \cdot S$, with $S := \frac{1}{L_1^{s_1}\cdots L_n^{s_n}} \in \mathcal{S}$, such that

- h is holomorphic at zero i.e. $h \in \mathcal{M}_+(\mathbb{C}^k)$;
- $\ell_1, \cdots, \ell_m, L_1, \cdots, L_n$ are linearly independent and $\langle \ell_1, \cdots, \ell_m \rangle \perp^Q \langle L_1, \cdots, L_n \rangle$ (locality independence).

They generate the subspace $\mathcal{M}_-^Q(\mathbb{C}^k) \subset \mathcal{M}(\mathbb{C}^k)$.

Laurent expansions (L. Guo, S.-P., B. Zhang [PJM 2020])

Second ingredient for the proof: Laurent expansions

Polar germs

A Q-**polar germ** in $\mathcal{M}(\mathbb{C}^k)$: $P := \frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}} = h\cdot S$, with $S := \frac{1}{L_1^{s_1}\cdots L_n^{s_n}} \in \mathcal{S}$, such that

- h is holomorphic at zero i.e. $h \in \mathcal{M}_+(\mathbb{C}^k)$;
- $\ell_1,\cdots,\ell_m, L_1,\cdots,L_n$ are linearly independent and $\langle \ell_1,\cdots,\ell_m\rangle \perp^Q \langle L_1,\cdots,L_n\rangle$ (locality independence).

They generate the subspace $\mathcal{M}_-^Q(\mathbb{C}^k) \subset \mathcal{M}(\mathbb{C}^k)$.

Laurent expansions (L. Guo, S.-P., B. Zhang [PJM 2020])

Given a meromorphic germ $f \in \mathcal{M}(\mathbb{C}^k)$, there exists a finite set of polar germs

$$\mathcal{M}_-^Q(\mathbb{C}^k) \ni \left\{ h_j \cdot S_j = \frac{h_j}{L_{j1}^{s_{j1}}\cdots L_{jn}^{s_{jn_j}}} \right\}_{j\in J}$$

Second ingredient for the proof: Laurent expansions

Polar germs

A Q-**polar germ** in $\mathcal{M}(\mathbb{C}^k)$: $P := \frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}} = h\cdot S$, with $S := \frac{1}{L_1^{s_1}\cdots L_n^{s_n}} \in \mathcal{S}$, such that

- h is holomorphic at zero i.e. $h \in \mathcal{M}_+(\mathbb{C}^k)$;
- $\ell_1,\cdots,\ell_m$, $L_1,\cdots,L_n$ are linearly independent and $\langle \ell_1,\cdots,\ell_m\rangle \perp^Q \langle L_1,\cdots,L_n\rangle$ (locality independence).

They generate the subspace $\mathcal{M}_-^Q(\mathbb{C}^k) \subset \mathcal{M}(\mathbb{C}^k)$.

Laurent expansions (L. Guo, S.-P., B. Zhang [PJM 2020])

Given a meromorphic germ $f \in \mathcal{M}(\mathbb{C}^k)$, there exists a finite set of polar germs

$\mathcal{M}_-^Q(\mathbb{C}^k) \ni \left\{h_j\cdot S_j = \frac{h_j}{L_{j1}^{s_{j1}}\cdots L_{jn}^{s_{jn_j}}}\right\}_{j\in J}$ i) that are properly positioned,

Second ingredient for the proof: Laurent expansions

Polar germs

A Q-**polar germ** in $\mathcal{M}(\mathbb{C}^k)$: $P := \frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}} = h\cdot S$, with $S := \frac{1}{L_1^{s_1}\cdots L_n^{s_n}} \in \mathcal{S}$, such that

- h is holomorphic at zero i.e. $h \in \mathcal{M}_+(\mathbb{C}^k)$;
- $\ell_1,\cdots,\ell_m, L_1,\cdots,L_n$ are linearly independent and $\langle \ell_1,\cdots,\ell_m\rangle \perp^Q \langle L_1,\cdots,L_n\rangle$ (locality independence).

They generate the subspace $\mathcal{M}_-^Q(\mathbb{C}^k) \subset \mathcal{M}(\mathbb{C}^k)$.

Laurent expansions (L. Guo, S.-P., B. Zhang [PJM 2020])

Given a meromorphic germ $f \in \mathcal{M}(\mathbb{C}^k)$, there exists a finite set of polar germs $\mathcal{M}_-^Q(\mathbb{C}^k) \ni \left\{h_j\cdot S_j = \frac{h_j}{L_{j1}^{s_{j1}}\cdots L_{jn}^{s_{jn_j}}}\right\}_{j\in J}$ i) that are properly positioned, ii) whose denominators are pairwise not proportional, and a holomorphic germ h, such that $f = h + \left[\sum_{j\in J} h_j\cdot S_j\right]$. In particular

$\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-^Q = \langle \mathcal{S}\rangle^{\perp^Q}{}_{\mathcal{M}_+}$ is a $\perp^Q$-locality $\mathcal{M}_+$-module generated by $\mathcal{S}$.

Second ingredient for the proof: Laurent expansions

Polar germs

A Q-**polar germ** in $\mathcal{M}(\mathbb{C}^k)$: $P := \frac{h(\ell_1,\cdots,\ell_m)}{L_1^{s_1}\cdots L_n^{s_n}} = h \cdot S$, with $S := \frac{1}{L_1^{s_1}\cdots L_n^{s_n}} \in \mathcal{S}$, such that

- h is holomorphic at zero i.e. $h \in \mathcal{M}_+(\mathbb{C}^k)$;
- $\ell_1, \cdots, \ell_m, L_1, \cdots, L_n$ are linearly independent and $\langle \ell_1, \cdots, \ell_m \rangle \perp^Q \langle L_1, \cdots, L_n \rangle$ (locality independence).

They generate the subspace $\mathcal{M}_-^Q(\mathbb{C}^k) \subset \mathcal{M}(\mathbb{C}^k)$.

Laurent expansions (L. Guo, S.-P., B. Zhang [PJM 2020])

Given a meromorphic germ $f \in \mathcal{M}(\mathbb{C}^k)$, there exists a finite set of polar germs $\mathcal{M}_-^Q(\mathbb{C}^k) \ni \left\{ h_j \cdot S_j = \frac{h_j}{L_{j1}^{s_{j1}}\cdots L_{jn}^{s_{jn_j}}} \right\}_{j\in J}$ i) that are properly positioned, ii) whose denominators are pairwise not proportional, and a holomorphic germ h, such that $f = h + \left[\sum_{j\in J} h_j \cdot S_j\right]$. In particular $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-^Q = \langle \mathcal{S} \rangle^{\perp^Q}{}_{\mathcal{M}_+}$ is a $\perp^Q$-locality $\mathcal{M}_+$-module generated by $\mathcal{S}$.

The decomposition is not unique $\frac{1}{L_1 L_2} = \frac{1}{L_1(L_1+L_2)} + \frac{1}{L_2(L_1+L_2)}$.

Third ingredient for the proof: **a Galois group of locality algebras**

Third ingredient for the proof: **a Galois group of locality algebras**

Let (A, m_A) be a $\top_A$- locality R-algebra, which is filtered $A = \cup_{p \in \mathbb{Z}_{\geq 0}} A_p$ and where A_p are $\top_A$- locality R-modules.

Third ingredient for the proof: **a Galois group of locality algebras**

Let (A, m_A) be a $\top_A$- locality R-algebra, which is filtered $A = \cup_{p\in\mathbb{Z}_{\geq 0}} A_p$ and where A_p are $\top_A$- locality R-modules.

Locality algebra morphisms

Third ingredient for the proof: a Galois group of locality algebras

Let (A, m_A) be a $\top_A$- locality R-algebra, which is filtered $A = \cup_{p\in\mathbb{Z}_{\geq 0}} A_p$ and where A_p are $\top_A$- locality R-modules.

Locality algebra morphisms

$f : A \longrightarrow A$ linear such that $f(r\, a) = r\, f(a)$ if $r \in a^{\top_A} \cap R$ ($\top_A$- loc. R-module morph.)

Third ingredient for the proof: **a Galois group of locality algebras**

Let (A, m_A) be a $\top_A$- locality R-algebra, which is filtered $A = \cup_{p \in \mathbb{Z}_{\geq 0}} A_p$ and where A_p are $\top_A$- locality R-modules.

Locality algebra morphisms

$f : A \to A$ linear such that $f(r\, a) = r\, f(a)$ if $r \in a^{\top_A} \cap R$ ($\top_A$- loc. R-module morph.) and $f \circ m_A(\top_A) = m_A \circ (f \otimes f) \circ \top_A$ ($\top_A$-loc. mult).

Galois group of A

Third ingredient for the proof: a Galois group of locality algebras

Let (A, m_A) be a $\top_A$- locality R-algebra, which is filtered $A = \cup_{p \in \mathbb{Z}_{\geq 0}} A_p$ and where A_p are $\top_A$- locality R-modules.

Locality algebra morphisms

$f : A \longrightarrow A$ linear such that $f(r\, a) = r\, f(a)$ if $r \in a^{\top_A} \cap R$ ($\top_A$- loc. R-module morph.) and $f \circ m_A(\top_A) = m_A \circ (f \otimes f) \circ \top_A$ ($\top_A$-loc. mult).

Galois group of A

The Galois group $\mathrm{Gal}^{\top_A}(A/R)$ of A is the set of

Third ingredient for the proof: a Galois group of locality algebras

Let (A, m_A) be a $\top_A$- locality R-algebra, which is filtered $A = \cup_{p \in \mathbb{Z}_{\geq 0}} A_p$ and where A_p are $\top_A$- locality R-modules.

Locality algebra morphisms

$f : A \to A$ linear such that $f(r\,a) = r\,f(a)$ if $r \in a^{\top_A} \cap R$ ($\top_A$- loc. R-module morph.) and $f \circ m_A(\top_A) = m_A \circ (f \otimes f) \circ \top_A$ ($\top_A$-loc. mult).

Galois group of A

The Galois group $\mathrm{Gal}^{\top_A}(A/R)$ of A is the set of $\top_A$- locality algebra isomorphisms of A over R compatible with the filtration,

Third ingredient for the proof: a Galois group of locality algebras

Let (A, m_A) be a $\top_A$- locality R-algebra, which is filtered $A = \cup_{p\in\mathbb{Z}_{\geq 0}} A_p$ and where A_p are $\top_A$- locality R-modules.

Locality algebra morphisms

$f : A \to A$ linear such that $f(r\,a) = r\,f(a)$ if $r \in a^{\top_A} \cap R$ ($\top_A$- loc. R-module morph.) and $f \circ m_A(\top_A) = m_A \circ (f \otimes f) \circ \top_A$ ($\top_A$-loc. mult).

Galois group of A

The Galois group $\mathrm{Gal}^{\top_A}(A/R)$ of A is the set of $\top_A$- locality algebra isomorphisms of A over R compatible with the filtration, that is

$$\mathrm{Gal}^{\top_A}(A/R) = \{T \in \mathrm{Aut}^{\top_A}(A) \mid T|_R = \mathrm{Id}, \quad T(A_p) \subset A_p\}.$$

The case of freely generated locality-algebras

Third ingredient for the proof: a Galois group of locality algebras

Let (A, m_A) be a $\top_A$- locality R-algebra, which is filtered $A = \cup_{p\in\mathbb{Z}_{\geq 0}} A_p$ and where A_p are $\top_A$- locality R-modules.

Locality algebra morphisms

$f : A \to A$ linear such that $f(r\,a) = r\,f(a)$ if $r \in a^{\top_A} \cap R$ ($\top_A$- loc. R-module morph.) and $f \circ m_A(\top_A) = m_A \circ (f \otimes f) \circ \top_A$ ($\top_A$-loc. mult).

Galois group of A

The Galois group $\mathrm{Gal}^{\top_A}(A/R)$ of A is the set of $\top_A$- locality algebra isomorphisms of A over R compatible with the filtration, that is

$$\mathrm{Gal}^{\top_A}(A/R) = \{T \in \mathrm{Aut}^{\top_A}(A) \mid T|_R = \mathrm{Id}, \quad T(A_p) \subset A_p\}.$$

The case of freely generated locality-algebras

If $A = \langle S \rangle_R^{\top_A}$ is a freely generated locality-algebra, then

Third ingredient for the proof: a Galois group of locality algebras

Let (A, m_A) be a $\top_A$- locality R-algebra, which is filtered $A = \cup_{p \in \mathbb{Z}_{\geq 0}} A_p$ and where A_p are $\top_A$- locality R-modules.

Locality algebra morphisms

$f : A \to A$ linear such that $f(r\,a) = r\,f(a)$ if $r \in a^{\top_A} \cap R$ ($\top_A$- loc. R-module morph.) and $f \circ m_A(\top_A) = m_A \circ (f \otimes f) \circ \top_A$ ($\top_A$-loc. mult).

Galois group of A

The Galois group $\mathrm{Gal}^{\top_A}(A/R)$ of A is the set of $\top_A$- locality algebra isomorphisms of A over R compatible with the filtration, that is

$$\mathrm{Gal}^{\top_A}(A/R) = \{T \in \mathrm{Aut}^{\top_A}(A) \mid T|_R = \mathrm{Id}, \quad T(A_p) \subset A_p\}.$$

The case of freely generated locality-algebras

If $A = \langle S \rangle_R^{\top_A}$ is a freely generated locality-algebra, then $T \in \mathrm{Gal}^{\top_A}(A/R)$ is uniquely determined by $T(s), s \in S$: $\quad T\left(\sum_{s \in S} r_s \cdot s\right) = \oplus_{s \in S} r_s \cdot T(s)$.

IV. Consequences for meromorphic germs

First consequence: a minimal subtraction scheme

First consequence: a minimal subtraction scheme

Let $\mathcal{A} := \mathcal{M}^{\bullet}(\mathbb{C}^\infty)$ be a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-subalgebra of $\left(\mathcal{M}(\mathbb{C}^\infty), \perp^Q\right)$.

First consequence: a minimal subtraction scheme

Let $\mathcal{A} := \mathcal{M}^{\bullet}(\mathbb{C}^\infty)$ be a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-subalgebra of $\left(\mathcal{M}(\mathbb{C}^\infty), \perp^Q\right)$. It is therefore also a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-module.

First consequence: a minimal subtraction scheme

Let $\mathcal{A} := \mathcal{M}^\bullet(\mathbb{C}^\infty)$ be a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-subalgebra of $\left(\mathcal{M}(\mathbb{C}^\infty), \perp^Q\right)$. It is therefore also a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-module.

Q -locality evaluators

A Q-locality evaluator on $\mathcal{A}$ is an $\mathcal{M}_+(\mathbb{C}^\infty)$-linear form $\mathcal{E} : \mathcal{A} \to \mathbb{C}$ on the locality $\mathcal{M}_+(\mathbb{C}^\infty)$-module,

First consequence: a minimal subtraction scheme

Let $\mathcal{A} := \mathcal{M}^\bullet(\mathbb{C}^\infty)$ be a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-subalgebra of $\left(\mathcal{M}(\mathbb{C}^\infty), \perp^Q\right)$. It is therefore also a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-module.

Q -locality evaluators

A Q-locality evaluator on $\mathcal{A}$ is an $\mathcal{M}_+(\mathbb{C}^\infty)$-linear form $\mathcal{E} : \mathcal{A} \to \mathbb{C}$ on the locality $\mathcal{M}_+(\mathbb{C}^\infty)$-module, compatible with the filtration, which is **partially multiplicative**:

First consequence: a minimal subtraction scheme

Let $\mathcal{A} := \mathcal{M}^\bullet(\mathbb{C}^\infty)$ be a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-subalgebra of $\left(\mathcal{M}(\mathbb{C}^\infty), \perp^Q\right)$. It is therefore also a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-module.

Q -locality evaluators

A Q-locality evaluator on $\mathcal{A}$ is an $\mathcal{M}_+(\mathbb{C}^\infty)$-linear form $\mathcal{E} : \mathcal{A} \to \mathbb{C}$ on the locality $\mathcal{M}_+(\mathbb{C}^\infty)$-module, compatible with the filtration, which is **partially multiplicative**:

$$(\text{locality/independence}) \quad \left(f \perp^Q g\right) \Longrightarrow \left(\mathcal{E}(f\,g) = \mathcal{E}(f)\,\mathcal{E}(g)\right).$$

Multi variable minimal subtraction: $\mathrm{ev}_0 \circ \pi_+{}^Q$

(N. Berline, M. Vergne 2015) Q-orthogonal splitting:

First consequence: a minimal subtraction scheme

Let $\mathcal{A} := \mathcal{M}^\bullet(\mathbb{C}^\infty)$ be a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-subalgebra of $\left(\mathcal{M}(\mathbb{C}^\infty), \perp^Q\right)$. It is therefore also a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-module.

Q -locality evaluators

A Q-locality evaluator on $\mathcal{A}$ is an $\mathcal{M}_+(\mathbb{C}^\infty)$-linear form $\mathcal{E} : \mathcal{A} \to \mathbb{C}$ on the locality $\mathcal{M}_+(\mathbb{C}^\infty)$-module, compatible with the filtration, which is **partially multiplicative**:

$$(\text{locality/independence}) \quad \left(f \perp^Q g\right) \Longrightarrow \left(\mathcal{E}(f\,g) = \mathcal{E}(f)\,\mathcal{E}(g)\right).$$

Multi variable minimal subtraction: $\mathrm{ev}_0 \circ \pi_+{}^Q$

(N. Berline, M. Vergne 2015) Q-orthogonal splitting:

$\mathcal{M}(\mathbb{C}^\infty) = \mathcal{M}_+(\mathbb{C}^\infty) \oplus^{\perp^Q} \mathcal{M}_-{}^Q(\mathbb{C}^\infty)$ (also follows from the Laurent decomposition)

First consequence: a minimal subtraction scheme

Let $\mathcal{A} := \mathcal{M}^{\bullet}(\mathbb{C}^\infty)$ be a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-subalgebra of $\left(\mathcal{M}(\mathbb{C}^\infty), \perp^Q\right)$. It is therefore also a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-module.

Q -locality evaluators

A Q-locality evaluator on $\mathcal{A}$ is an $\mathcal{M}_+(\mathbb{C}^\infty)$-linear form $\mathcal{E} : \mathcal{A} \to \mathbb{C}$ on the locality $\mathcal{M}_+(\mathbb{C}^\infty)$-module, compatible with the filtration, which is **partially multiplicative**:

$$(\text{locality/independence}) \quad \left(f \perp^Q g\right) \Longrightarrow \left(\mathcal{E}(f\,g) = \mathcal{E}(f)\,\mathcal{E}(g)\right).$$

Multi variable minimal subtraction: $\mathrm{ev}_0 \circ \pi_+{}^Q$

(N. Berline, M. Vergne 2015) Q-orthogonal splitting:

$\mathcal{M}(\mathbb{C}^\infty) = \mathcal{M}_+(\mathbb{C}^\infty) \oplus^{\perp^Q} \mathcal{M}_-{}^Q(\mathbb{C}^\infty)$ (also follows from the Laurent decomposition)

$\Longrightarrow \pi_+{}^Q : \mathcal{M}(\mathbb{C}^\infty) \to \mathcal{M}_+(\mathbb{C}^\infty)$ is a locality morphism;

First consequence: a minimal subtraction scheme

Let $\mathcal{A} := \mathcal{M}^\bullet(\mathbb{C}^\infty)$ be a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-subalgebra of $\left(\mathcal{M}(\mathbb{C}^\infty), \perp^Q\right)$. It is therefore also a locality $\mathcal{M}_+(\mathbb{C}^\infty)$-module.

Q -locality evaluators

A Q-locality evaluator on $\mathcal{A}$ is an $\mathcal{M}_+(\mathbb{C}^\infty)$-linear form $\mathcal{E} : \mathcal{A} \to \mathbb{C}$ on the locality $\mathcal{M}_+(\mathbb{C}^\infty)$-module, compatible with the filtration, which is **partially multiplicative**:

$$(\text{locality/independence}) \quad \left(f \perp^Q g\right) \Longrightarrow \left(\mathcal{E}(f\,g) = \mathcal{E}(f)\,\mathcal{E}(g)\right).$$

Multi variable minimal subtraction: $\mathrm{ev}_0 \circ \pi_+{}^Q$

(N. Berline, M. Vergne 2015) Q-orthogonal splitting:

$\mathcal{M}(\mathbb{C}^\infty) = \mathcal{M}_+(\mathbb{C}^\infty) \oplus^{\perp^Q} \mathcal{M}_-Q(\mathbb{C}^\infty)$ (also follows from the Laurent decomposition)

$\Longrightarrow \pi_+{}^Q : \mathcal{M}(\mathbb{C}^\infty) \to \mathcal{M}_+(\mathbb{C}^\infty)$ is a locality morphism;

$\Longrightarrow \mathcal{E}^Q = \mathrm{ev}_0 \circ \pi_+{}^Q$ is a Q-locality evaluator on $\mathcal{M}(\mathbb{C}^\infty)$.

Second consequence: polynomial algebras of germs

From now on, Q is the canonical inner product on $\mathbb{R}^\infty$ and we write $\perp$.

Second consequence: polynomial algebras of germs

From now on, Q is the canonical inner product on $\mathbb{R}^\infty$ and we write $\perp$.

Meromorphic germs with prescribed pole type

Second consequence: polynomial algebras of germs

From now on, Q is the canonical inner product on $\mathbb{R}^\infty$ and we write $\perp$.

Meromorphic germs with prescribed pole type

[Guo, S.P., Zhang, 2021] The $\perp$ -locality algebras $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ of meromorphic germs with Chen, resp. Feynman type linear poles, are freely generated by the polynomial generators

Second consequence: polynomial algebras of germs

From now on, Q is the canonical inner product on $\mathbb{R}^\infty$ and we write $\perp$.

Meromorphic germs with prescribed pole type

[Guo, S.P., Zhang, 2021] The $\perp$ -locality algebras $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ of meromorphic germs with Chen, resp. Feynman type linear poles, are freely generated by the polynomial generators

1. the set $\mathcal{S}^{\mathrm{Chen}}$ of **Chen fractions** of the form $\frac{1}{z_{i_1}^{s_1}(z_{i_1}+z_{i_2})^{s_2}\cdots(z_{i_1}+z_{i_2}+\cdots+z_{i_k})^{s_k}}$ with $s_i \in \mathbb{Z}_{\geq 0}$ and distinct $i_1, \cdots, i_k \in \mathbb{Z}_{\geq 0}$;

Second consequence: polynomial algebras of germs

From now on, Q is the canonical inner product on $\mathbb{R}^\infty$ and we write $\perp$.

Meromorphic germs with prescribed pole type

[Guo, S.P., Zhang, 2021] The $\perp$ -locality algebras $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ of meromorphic germs with Chen, resp. Feynman type linear poles, are freely generated by the polynomial generators

1. the set $\mathcal{S}^{\mathrm{Chen}}$ of **Chen fractions** of the form $\frac{1}{z_{i_1}^{s_1}(z_{i_1}+z_{i_2})^{s_2}\cdots(z_{i_1}+z_{i_2}+\cdots+z_{i_k})^{s_k}}$ with $s_i \in \mathbb{Z}_{\geq 0}$ and distinct $i_1, \cdots, i_k \in \mathbb{Z}_{\geq 0}$;
2. resp. the set $\mathcal{S}^{\mathrm{Feyn}}$ of **Feynman fractions** of the form $\prod_{\ell=1}^n \frac{1}{\left(\sum_{j\in I_\ell} z_j\right)^{s_{I_\ell}}}$ with $s_i \in \mathbb{Z}_{\geq 0}$ and $I_1 \subset \cdots \subset I_n \subseteq \{1, \cdots, k\}$.

Second consequence: polynomial algebras of germs

From now on, Q is the canonical inner product on $\mathbb{R}^\infty$ and we write $\perp$.

Meromorphic germs with prescribed pole type

[Guo, S.P., Zhang, 2021] The $\perp$ -locality algebras $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ of meromorphic germs with Chen, resp. Feynman type linear poles, are freely generated by the polynomial generators

1. the set $\mathcal{S}^{\rm Chen}$ of **Chen fractions** of the form $\frac{1}{z_{i_1}^{s_1}(z_{i_1}+z_{i_2})^{s_2}\cdots(z_{i_1}+z_{i_2}+\cdots+z_{i_k})^{s_k}}$ with $s_i \in \mathbb{Z}_{\geq 0}$ and distinct $i_1, \cdots, i_k \in \mathbb{Z}_{\geq 0}$;
2. resp. the set $\mathcal{S}^{\rm Feyn}$ of **Feynman fractions** of the form $\prod_{\ell=1}^{n} \frac{1}{\left(\sum_{j\in I_\ell} z_j\right)^{s_{I_\ell}}}$ with $s_i \in \mathbb{Z}_{\geq 0}$ and $I_1 \subset \cdots \subset I_n \subseteq \{1, \cdots, k\}$.

In particular, $\mathcal{M}^{Chen} = \langle \mathcal{S}^{\rm Chen} \rangle_{\mathcal{M}_+}{}^{\perp}$, resp. $\mathcal{M}^{Feyn} = \langle \mathcal{S}^{\rm Feyn} \rangle_{\mathcal{M}_+}{}^{\perp}$

Second consequence: polynomial algebras of germs

From now on, Q is the canonical inner product on $\mathbb{R}^\infty$ and we write $\perp$.

Meromorphic germs with prescribed pole type

[Guo, S.P., Zhang, 2021] The $\perp$ -locality algebras $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ of meromorphic germs with Chen, resp. Feynman type linear poles, are freely generated by the polynomial generators

1. the set $\mathcal{S}^{\rm Chen}$ of **Chen fractions** of the form $\frac{1}{z_{i_1}^{s_1}(z_{i_1}+z_{i_2})^{s_2}\cdots(z_{i_1}+z_{i_2}+\cdots+z_{i_k})^{s_k}}$ with $s_i \in \mathbb{Z}_{\geq 0}$ and distinct $i_1, \cdots, i_k \in \mathbb{Z}_{\geq 0}$;
2. resp. the set $\mathcal{S}^{\rm Feyn}$ of **Feynman fractions** of the form $\prod_{\ell=1}^n \frac{1}{\left(\sum_{j\in I_\ell} z_j\right)^{s_{I_\ell}}}$ with $s_i \in \mathbb{Z}_{\geq 0}$ and $I_1 \subset \cdots \subset I_n \subseteq \{1, \cdots, k\}$.

In particular, $\mathcal{M}^{Chen} = \langle \mathcal{S}^{\rm Chen}\rangle_{\mathcal{M}_+}{}^{\perp}$, resp. $\mathcal{M}^{Feyn} = \langle \mathcal{S}^{\rm Feyn}\rangle_{\mathcal{M}_+}{}^{\perp}$ are freely generated as $\perp$-locality $\mathcal{M}_+$- modules.

The proof uses a locality version of Radford's theorem [Journal of Algebra 1979], which says that Lyndon words are polynomial generators of a shuffle algebra of words.

Second consequence: polynomial algebras of germs

From now on, Q is the canonical inner product on $\mathbb{R}^\infty$ and we write $\perp$.

Meromorphic germs with prescribed pole type

[Guo, S.P., Zhang, 2021] The $\perp$ -locality algebras $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ of meromorphic germs with Chen, resp. Feynman type linear poles, are freely generated by the polynomial generators

1. the set $\mathcal{S}^{\rm Chen}$ of **Chen fractions** of the form $\frac{1}{z_{i_1}^{s_1}(z_{i_1}+z_{i_2})^{s_2}\cdots(z_{i_1}+z_{i_2}+\cdots+z_{i_k})^{s_k}}$ with $s_i \in \mathbb{Z}_{\geq 0}$ and distinct $i_1, \cdots, i_k \in \mathbb{Z}_{\geq 0}$;
2. resp. the set $\mathcal{S}^{\rm Feyn}$ of **Feynman fractions** of the form $\prod_{\ell=1}^{n} \frac{1}{\left(\sum_{j\in I_\ell} z_j\right)^{s_{I_\ell}}}$ with $s_i \in \mathbb{Z}_{\geq 0}$ and $I_1 \subset \cdots \subset I_n \subseteq \{1, \cdots, k\}$.

In particular, $\mathcal{M}^{Chen} = \langle \mathcal{S}^{\rm Chen} \rangle_{\mathcal{M}_+}{}^{\perp}$, resp. $\mathcal{M}^{Feyn} = \langle \mathcal{S}^{\rm Feyn} \rangle_{\mathcal{M}_+}{}^{\perp}$ are freely generated as $\perp$-locality $\mathcal{M}_+$- modules.

The proof uses a locality version of Radford's theorem [Journal of Algebra 1979], which says that Lyndon words are polynomial generators of a shuffle algebra of words.

The Galois group

The Galois group $\mathrm{Gal}^{\perp}(\mathcal{M}^\bullet/\mathcal{M}_+) = \{T \in \mathrm{Aut}^{\perp^Q}(\mathcal{M}^\bullet) \mid T|_{\mathcal{M}_+} = \mathrm{Id}, \;\; T(\mathcal{M}_p) \subset \mathcal{M}_p\}$ for $\mathcal{M}^{Chen}$ and $\mathcal{M}^{Feyn}$

Second consequence: polynomial algebras of germs

From now on, Q is the canonical inner product on $\mathbb{R}^\infty$ and we write $\perp$.

Meromorphic germs with prescribed pole type

[Guo, S.P., Zhang, 2021] The $\perp$ -locality algebras $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ of meromorphic germs with Chen, resp. Feynman type linear poles, are freely generated by the polynomial generators

1. the set $\mathcal{S}^{\mathrm{Chen}}$ of **Chen fractions** of the form $\frac{1}{z_{i_1}^{s_1}(z_{i_1}+z_{i_2})^{s_2}\cdots(z_{i_1}+z_{i_2}+\cdots+z_{i_k})^{s_k}}$ with $s_i \in \mathbb{Z}_{\geq 0}$ and distinct $i_1, \cdots, i_k \in \mathbb{Z}_{\geq 0}$;
2. resp. the set $\mathcal{S}^{\mathrm{Feyn}}$ of **Feynman fractions** of the form $\prod_{\ell=1}^n \frac{1}{\left(\sum_{j\in I_\ell} z_j\right)^{s_{I_\ell}}}$ with $s_i \in \mathbb{Z}_{\geq 0}$ and $I_1 \subset \cdots \subset I_n \subseteq \{1, \cdots, k\}$.

In particular, $\mathcal{M}^{Chen} = \langle \mathcal{S}^{\mathrm{Chen}} \rangle_{\mathcal{M}_+}{}^{\perp}$, resp. $\mathcal{M}^{Feyn} = \langle \mathcal{S}^{\mathrm{Feyn}} \rangle_{\mathcal{M}_+}{}^{\perp}$ are freely generated as $\perp$-locality $\mathcal{M}_+$- modules.

The proof uses a locality version of Radford's theorem [Journal of Algebra 1979], which says that Lyndon words are polynomial generators of a shuffle algebra of words.

The Galois group

The Galois group $\mathrm{Gal}^{\perp}(\mathcal{M}^\bullet/\mathcal{M}_+) = \{T \in \mathrm{Aut}^{\perp^Q}(\mathcal{M}^\bullet) \mid T|_{\mathcal{M}_+} = \mathrm{Id}, \quad T(\mathcal{M}_p) \subset \mathcal{M}_p\}$ for $\mathcal{M}^{Chen}$ and $\mathcal{M}^{Feyn}$ is defined by means of the filtration on $\mathcal{S}^\bullet$ by the order of the poles.

Final step of the proof

Final step of the proof

We have shown that $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ are freely generated $\perp$-locality $\mathcal{M}_+$-algebras $\mathcal{M}^\bullet := \langle \mathcal{S}^\bullet \rangle_{\mathcal{M}_+}{}^{\perp}$.

Final step of the proof

We have shown that $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ are freely generated $\perp$-locality $\mathcal{M}_+$-algebras $\mathcal{M}^\bullet := \langle \mathcal{S}^\bullet \rangle_{\mathcal{M}_+}{}^{\perp}$.

The case of freely generated locality-algebras

Final step of the proof

We have shown that $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ are freely generated $\perp$-locality $\mathcal{M}_+$-algebras $\mathcal{M}^\bullet := \langle \mathcal{S}^\bullet \rangle_{\mathcal{M}_+}{}^{\perp}$.

The case of freely generated locality-algebras

Since $\mathcal{M} = \langle \mathcal{S} \rangle_{\mathcal{M}_+}^{\perp}$ is a freely generated locality-algebra,

Final step of the proof

We have shown that $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ are freely generated $\perp$-locality $\mathcal{M}_+$-algebras $\mathcal{M}^\bullet := \langle \mathcal{S}^\bullet \rangle_{\mathcal{M}_+}{}^{\perp}$.

The case of freely generated locality-algebras

Since $\mathcal{M} = \langle \mathcal{S} \rangle_{\mathcal{M}_+}^{\perp}$ is a freely generated locality-algebra, $T \in \mathrm{Gal}^{\perp}(\mathcal{M}/\mathcal{M}_+)$ is uniquely determined by $\{T(S), S \in \mathcal{S}\}$: $T\left(\sum_{S \in \mathcal{S}} h_S \cdot S\right) = \sum_{S \in \mathcal{S}} h_S \cdot T(S)$.

$\perp$-locality evaluators

Final step of the proof

We have shown that $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ are freely generated $\perp$-locality $\mathcal{M}_+$-algebras $\mathcal{M}^\bullet := \langle \mathcal{S}^\bullet \rangle_{\mathcal{M}_+}{}^\perp$.

The case of freely generated locality-algebras

Since $\mathcal{M} = \langle \mathcal{S} \rangle^\perp_{\mathcal{M}_+}$ is a freely generated locality-algebra, $T \in \mathrm{Gal}^\perp(\mathcal{M}/\mathcal{M}_+)$ is uniquely determined by $\{T(S), S \in \mathcal{S}\}$: $\quad T\left(\sum_{S\in\mathcal{S}} h_S \cdot S\right) = \sum_{S\in\mathcal{S}} h_S \cdot T(S)$.

$\perp$-locality evaluators

Given a $\perp$-locality evaluator $\mathcal{E}$ on a freely generated algebra $\mathcal{M}^\bullet := \langle \mathcal{S}^\bullet \rangle_{\mathcal{M}_+}{}^\perp$ of meromorphic germs,

Final step of the proof

We have shown that $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ are freely generated $\perp$-locality $\mathcal{M}_+$-algebras $\mathcal{M}^\bullet := \langle \mathcal{S}^\bullet \rangle_{\mathcal{M}_+}{}^\perp$.

The case of freely generated locality-algebras

Since $\mathcal{M} = \langle \mathcal{S} \rangle^\perp_{\mathcal{M}_+}$ is a freely generated locality-algebra, $T \in \mathrm{Gal}^\perp(\mathcal{M}/\mathcal{M}_+)$ is uniquely determined by $\{T(S), S \in \mathcal{S}\}$: $\quad T\left(\sum_{S\in\mathcal{S}} h_S \cdot S\right) = \sum_{S\in\mathcal{S}} h_S \cdot T(S)$.

$\perp$-locality evaluators

Given a $\perp$-locality evaluator $\mathcal{E}$ on a freely generated algebra $\mathcal{M}^\bullet := \langle \mathcal{S}^\bullet \rangle_{\mathcal{M}_+}{}^\perp$ of meromorphic germs, the map

$$T_{\mathcal{E}} : S \mapsto S + \mathcal{E}(S)\, 1$$

defines an element of the Galois group $\mathrm{Gal}^\perp(\mathcal{M}/\mathcal{M}_+)$

Final step of the proof

We have shown that $\mathcal{M}^{Chen}$, resp. $\mathcal{M}^{Feyn}$ are freely generated $\perp$-locality $\mathcal{M}_+$-algebras $\mathcal{M}^\bullet := \langle \mathcal{S}^\bullet \rangle_{\mathcal{M}_+}{}^{\perp}$.

The case of freely generated locality-algebras

Since $\mathcal{M} = \langle \mathcal{S} \rangle_{\mathcal{M}_+}^{\perp}$ is a freely generated locality-algebra, $T \in \mathrm{Gal}^{\perp}(\mathcal{M}/\mathcal{M}_+)$ is uniquely determined by $\{T(S), S \in \mathcal{S}\}$: $T\left(\sum_{S\in\mathcal{S}} h_S \cdot S\right) = \sum_{S\in\mathcal{S}} h_S \cdot T(S)$.

$\perp$-locality evaluators

Given a $\perp$-locality evaluator $\mathcal{E}$ on a freely generated algebra $\mathcal{M}^\bullet := \langle \mathcal{S}^\bullet \rangle_{\mathcal{M}_+}{}^{\perp}$ of meromorphic germs, the map

$$T_{\mathcal{E}} : S \mapsto S + \mathcal{E}(S)\,1$$

defines an element of the Galois group $\mathrm{Gal}^{\perp}(\mathcal{M}/\mathcal{M}_+)$ and

$$\mathcal{E} = \underbrace{\mathrm{ev}_0 \circ \pi_+{}^{\perp}}_{\mathcal{E}^{\perp}\text{ minimal subtraction}} \circ \underbrace{T_{\mathcal{E}}}_{\text{Galois transformation}} .$$

Open questions

Open questions

Dependence on the choice of Q

- inner product: $Q \rightsquigarrow Q' = Q \circ (g \otimes g)$ with $g \in GL(\mathbb{R}^\infty)$;

Open questions

Dependence on the choice of Q

- inner product: $Q \rightsquigarrow Q' = Q \circ (g \otimes g)$ with $g \in GL(\mathbb{R}^\infty)$;
- evaluator: $\mathcal{E}^{\perp^Q} = \mathrm{ev}_0 \circ {\pi_+}^{\perp^Q} \rightsquigarrow \mathcal{E}^{\perp^{Q'}} = \mathcal{E}^{\perp^Q} \circ g$;

Open questions

Dependence on the choice of Q

- inner product: $Q \rightsquigarrow Q' = Q \circ (g \otimes g)$ with $g \in GL(\mathbb{R}^\infty)$;
- evaluator: $\mathcal{E}^{\perp^Q} = \mathrm{ev}_0 \circ \pi_+{}^{\perp^Q} \rightsquigarrow \mathcal{E}^{\perp^{Q'}} = \mathcal{E}^{\perp^Q} \circ g$;
- Galois group: $\mathrm{Gal}^{\perp^Q}(\mathcal{M}/\mathcal{M}_+) \rightsquigarrow \mathrm{Gal}^{\perp^{Q'}}(\mathcal{M}/\mathcal{M}_+)$

Open questions

Dependence on the choice of Q

- inner product: $Q \rightsquigarrow Q' = Q \circ (g \otimes g)$ with $g \in GL(\mathbb{R}^\infty)$;
- evaluator: $\mathcal{E}^{\perp^Q} = \mathrm{ev}_0 \circ \pi_+{}^{\perp^Q} \rightsquigarrow \mathcal{E}^{\perp^{Q'}} = \mathcal{E}^{\perp^Q} \circ g$;
- Galois group: $\mathrm{Gal}^{\perp^Q}(\mathcal{M}/\mathcal{M}_+) \rightsquigarrow \mathrm{Gal}^{\perp^{Q'}}(\mathcal{M}/\mathcal{M}_+)$

and beyond $\perp$-locality relations.

THANK YOU FOR YOUR ATTENTION!

- P. Clavier, L. Guo, B. Zhang and S. P., An algebraic formulation of the locality principle in renormalisation, *European Journal of Mathematics*, Volume 5 (2019) 356-394
- P. Clavier, L. Guo, B. Zhang and S. P., Renormalisation via locality morphisms, *Revista Colombiana de Matemáticas*, Volume 53 (2019) 113-141
- P. Clavier, L. Guo, B. Zhang and S. P., Renormalisation and locality: branched zeta values, in ”Algebraic Combinatorics, Resurgence, Moulds and Applications (Carma)” Vol. 2 ,Eds. F. Chapoton, F. Fauvet, C. Malvenuto, J.-Y. Thibon, Irma Lectures in Mathematics and Theoretical Physics **32**, *European Math. Soc.* (2020) 85–132
- P. Clavier, L. Guo, B. Zhang and S. P., Locality and renormalisation: universal properties and integrals on trees, *Journal of Mathematical Physics* **61**, 022301 (2020)
- L. Guo, B. Zhang and S. P., Renormalisation and the Euler-Maclaurin formula on cones, *Duke Math J.*, **166** (3) (2017) 537–571.
- L. Guo, B. Zhang and S. P., A conical approach to Laurent expansions for multivariate meromorphic germs with linear poles, *Pacific Journal of Mathematics* 307 (2020) 159–196.