Simulating spectral triples

Lisa Glaser

2nd February 2021

Plan for this talk:

- Motivation and Method
- The Heisenberg relation
- Painting a picture

universität wien

Talk based on work with Abel Stern
arXiv:1909.08054 and arXiv:1912.09227

Quantum Gravity on the computer

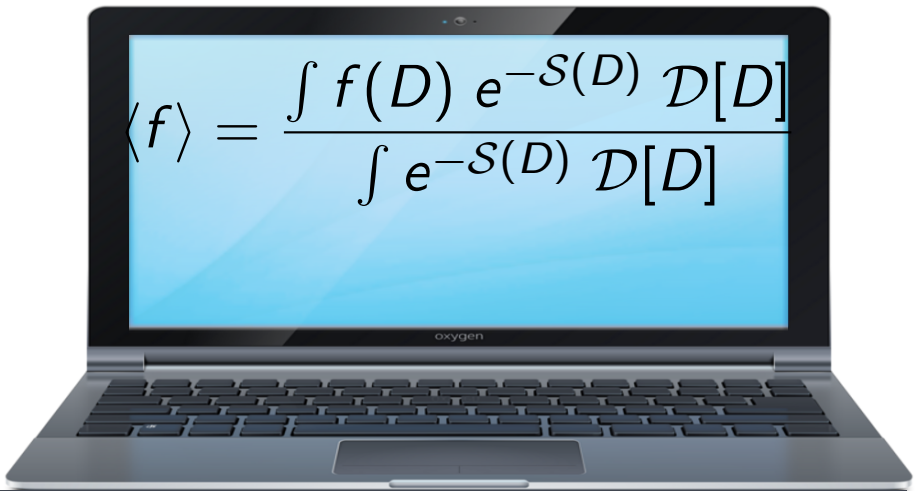
Computer simulations are powerful.
They can:

- Calculate the mass of the Proton (lattice QCD)
- Predict gravitational wave signatures of colliding neutron stars and black holes (numerical relativity)
- Maybe they can also help us understand Quantum Gravity?

Path Integral on the computer

Recipe:

- Define your theory
- Pick your observables
- Choose your algorithm



Ingredients:

- Geometry, here the Dirac operator D and measure $\mathcal{D}[D]$
- Functions of the geometry f
- Action $\mathcal{S}$ (in the physicists sense, an energy, part of the measure)

(Barrett, LG J.Phys. A49, 245001 (2016))

(LG J.Phys.A50 275201 (2017))

(Barrett, Druce, LG J.Phys. A52, 275203 (2019))

Monte Carlo Simulations

- Simulate a Path integral, use Monte Carlo Markov Chain to calculate averages
- Use Markov Chain to probe space of solutions to find an optimum. Only examine the solution with minimal value of something.

Monte Carlo Simulations

- Simulate a Path integral, use Monte Carlo Markov Chain to calculate averages
- Use Markov Chain to probe space of solutions to find an optimum. Only examine the solution with minimal value of something.

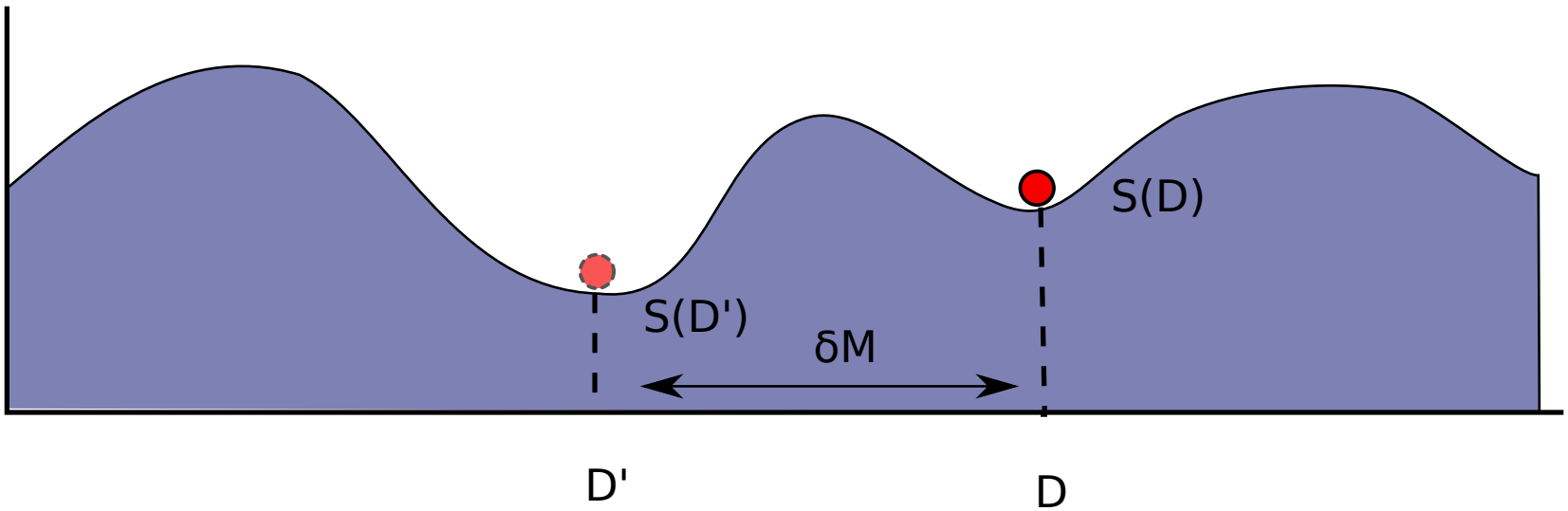
Note:

It is proven that the Metropolis algorithm will find the global optimum if sampled long enough.

Markov Chain Methods in one slide

The Metropolis Hastings algorithm

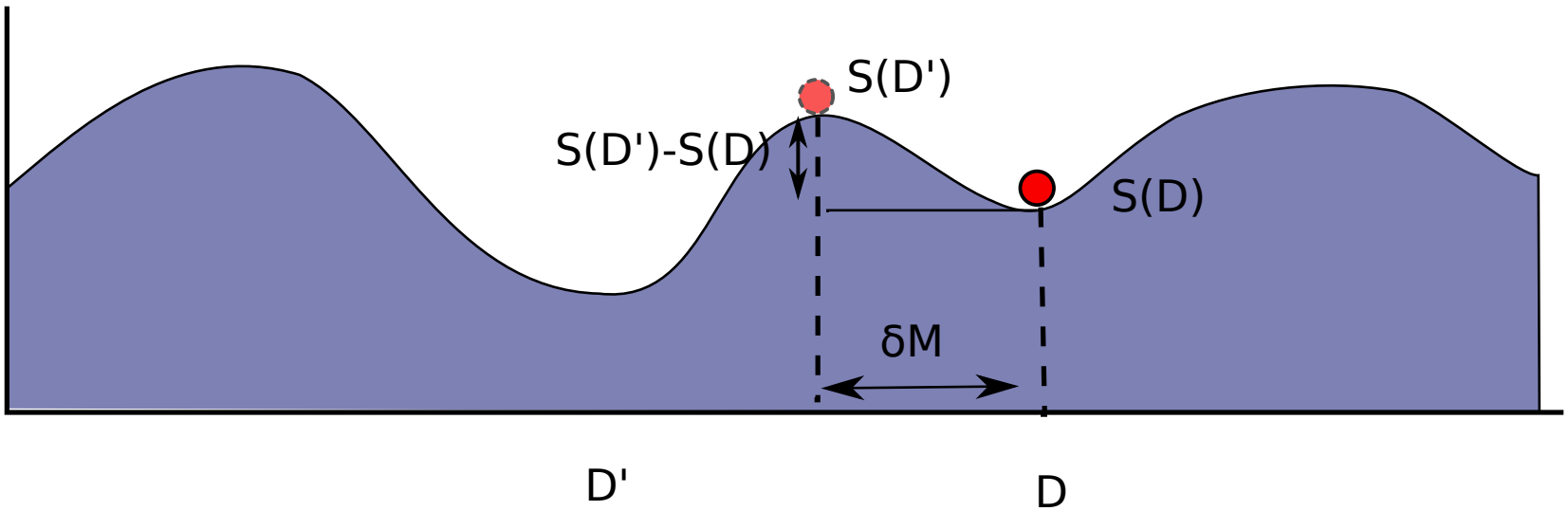
- propose new operator D'
  $D \to D' = D + \delta M$ with δM some small matrix
- if $S(D') < S(D)$ accept D' and add to the chain
- otherwise calculate $\exp\{-S(D') + S(D)\}$ & generate random uniform $p \in [0, 1]$
  - if $p < \exp\{-S(D') + S(D)\}$ accept D'
  - else add D to the chain again



Markov Chain Methods in one slide

The Metropolis Hastings algorithm

- propose new operator D'
  $D \to D' = D + \delta M$ with δM some small matrix
- if $S(D') < S(D)$ accept D' and add to the chain
- otherwise calculate $\exp\{-S(D') + S(D)\}$ & generate random uniform $p \in [0, 1]$
  - if $p < \exp\{-S(D') + S(D)\}$ accept D'
  - else add D to the chain again



Geometry as a spectral triple

$(\mathcal{A}, \mathcal{H}, D)$ Algebra $\mathcal{A}$ with action on a Hilbert space $\mathcal{H}$ and a Dirac operator D

Axioms of non-commutative geometry

1. The n-th characteristic value of the resolvent of D is $O(n^{-\frac{1}{p}})$.
2. $[[D, a], b] = 0 \forall a, b \in \mathcal{A}$ first order condition
3. For any $a \in \mathcal{A}$ both a and $[D, a]$ belong to the domain of δ^m , for any integer m where δ is the derivation: $\delta(T) = [|D|, T]$.
4. There exists a Hochschild cycle $c \in Z_p(\mathcal{A}, \mathcal{A})$ such that $\pi_D(c) = 1$ for p odd, while for p even, $\pi_D(c) = \gamma$ is a $\mathbb{Z}/2$ grading.
5. Viewed as an $\mathcal{A}$-module the space $\mathcal{H}_\infty = \bigcap DomD^m$ is finite and projective. Moreover the following equality defines a hermitian structure $(|)$ on this module: $\langle \xi, a\eta \rangle = \fint a(\xi|\eta)|D|^{-p}, \forall a \in \mathcal{A}, \forall \xi, \eta \in \mathcal{H}_\infty$

(A. Connes, Int.J.Geom.Meth.Mod.Phys. 5, 1215-1242 (2008))

(more detail e.g. A. Connes, Commun.Math.Phys. 182, 155-176 (1996))

A truncated system

$\mathcal{A}, D$ are infinite dimensional . . .

Truncate D

Replace the infinite D by a $n \times n$ matrix [1]

$$D \to P_n D P_n$$

with P_n a projector on the n smallest eigenvalues.

We assume that the finite D is a truncation of the infinite one, and that there are no small eigenvalues that we don't see.

[1] We could also use fuzzy spaces, for work on that see (J. W. Barrett, LG, J.Phys. A49, 245001 (2016))

A truncated system

$\mathcal{A}, D$ are infinite dimensional . . .

Truncate D

Replace the infinite D by a $n \times n$ matrix [1]

$$D \to P_n D P_n$$

with P_n a projector on the n smallest eigenvalues.

We assume that the finite D is a truncation of the infinite one, and that there are no small eigenvalues that we don't see.

Can now use computer simulations to understand the theory

[1] We could also use fuzzy spaces, for work on that see (J. W. Barrett, LG, J.Phys. A49, 245001 (2016))

Interlude: Operator system spectral triples

Operator systems: the formal version

Operator systems provide the mathematically rigorous version to our "just cutting of the spectral triple".
Algebra is replaced through an operator system $\mathcal{E} = P\mathcal{A}P$
The operator system spectral triple is $(\mathcal{E}, \mathcal{H}, D)$.

States are well defined $\Rightarrow$ distance measure can be generalized to:

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{E}} \{|\omega_1(a) - \omega_2(a)| : ||[D, a]|| \leq 1\}$$

(Connes,van Suijlekom arXiv:2004.14115, van Suijlekom arXiv:2005.08544)

Conditions on geometry

The one sided Heisenberg relation

$$\left\langle Y[D, Y]^d \right\rangle = \gamma$$

Where γ is a chirality and
$Y \in \mathcal{A} \otimes C_k$, $Y = \sum_i \Gamma_i Y^i$ with $\Gamma_i \in C_k$, and $Y^2 = \sum_i Y^i Y^i = 1$

Y is idempotent and Y^i are embedding maps for the sphere

Quanta of geometry

If D satisfies this equation & the axioms above the spectral triple is a union of non-commutative d-spheres (for infinite spectra).

(A.H. Chamseddine, A. Connes, V. Mukhanov, Phys.Rev.Lett. 114, 091302 (2015))

Heisenberg relations as a constraint

Can turn the one sided Heisenberg relation into a constraint for computer simulations

$$\left\| \left\langle Y[D, Y]^d \right\rangle - \gamma \right\|^2_{HS}$$

with $\|a_{ij}\|_{HS}$ the Hilbert-Schmidt norm (element wise norm)

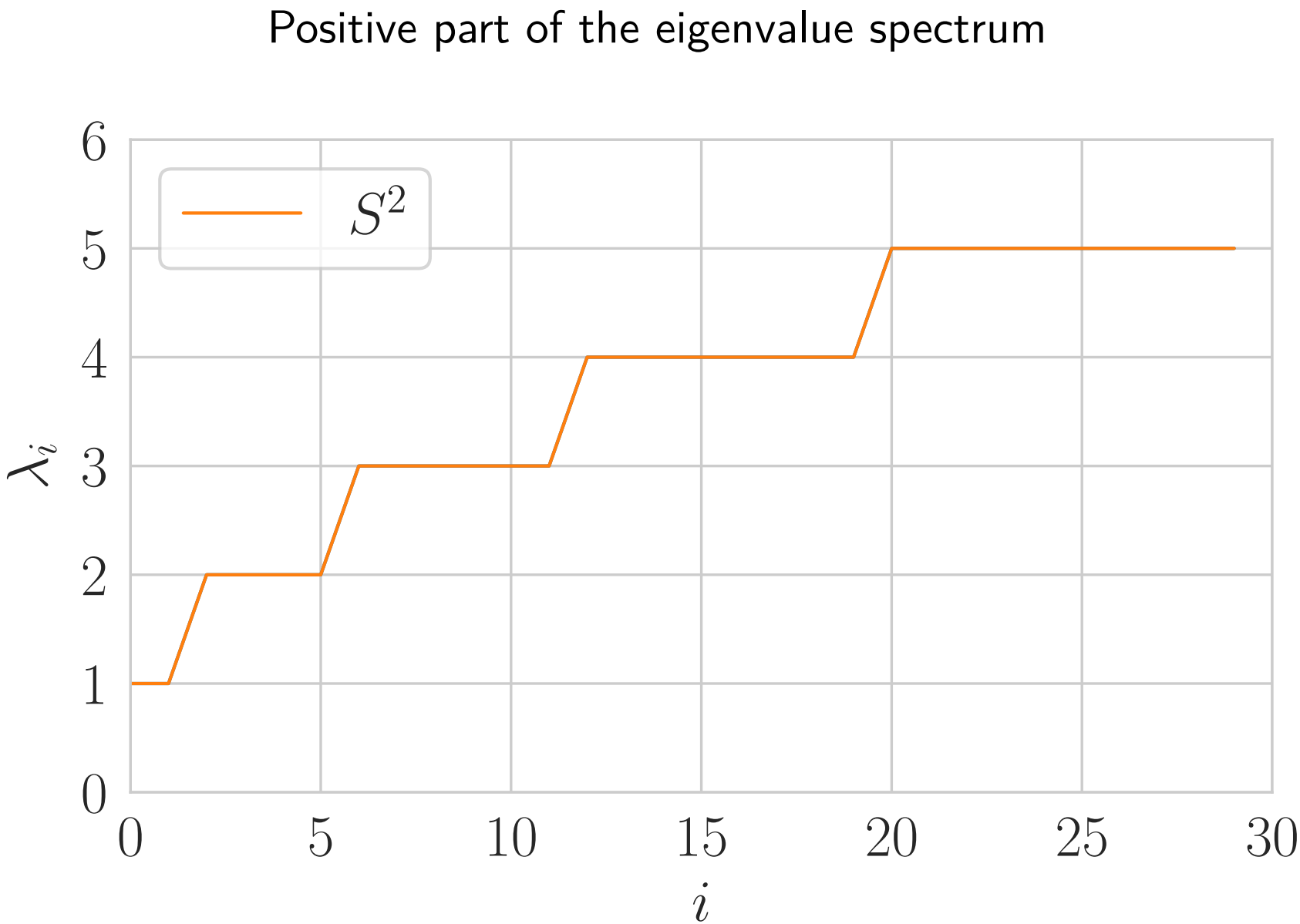
Motivation:

Using this as an action $\mathcal{S}$ in Monte Carlo simulations should force the spectral triples probed by the algorithm to be close to d-spheres

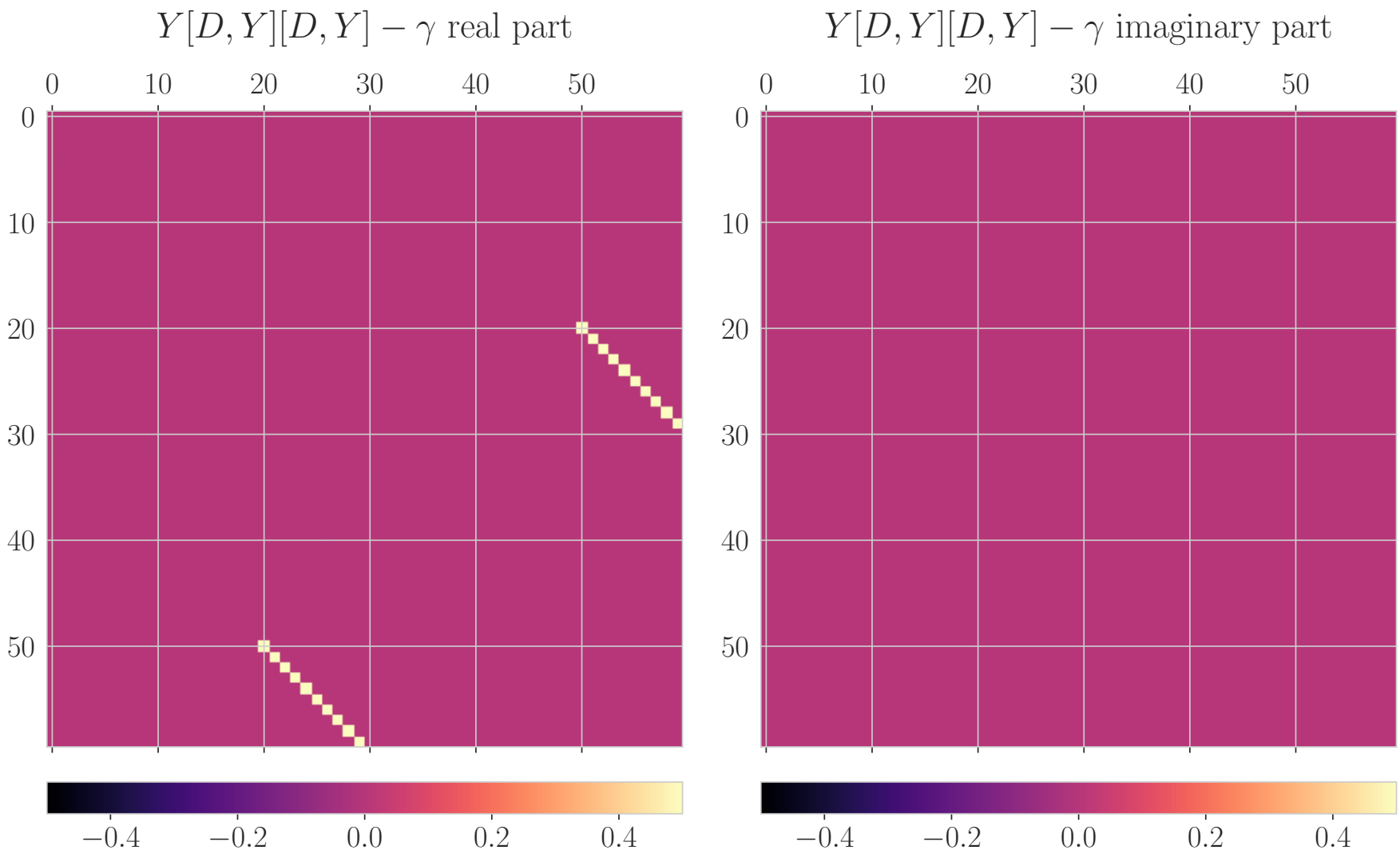
(A.H. Chamseddine, A. Connes, V. Mukhanov, Phys.Rev.Lett. 114, 091302 (2015))

(LG, A. Stern, W. van Suijlekom work in progress)

This is a (truncated) sphere



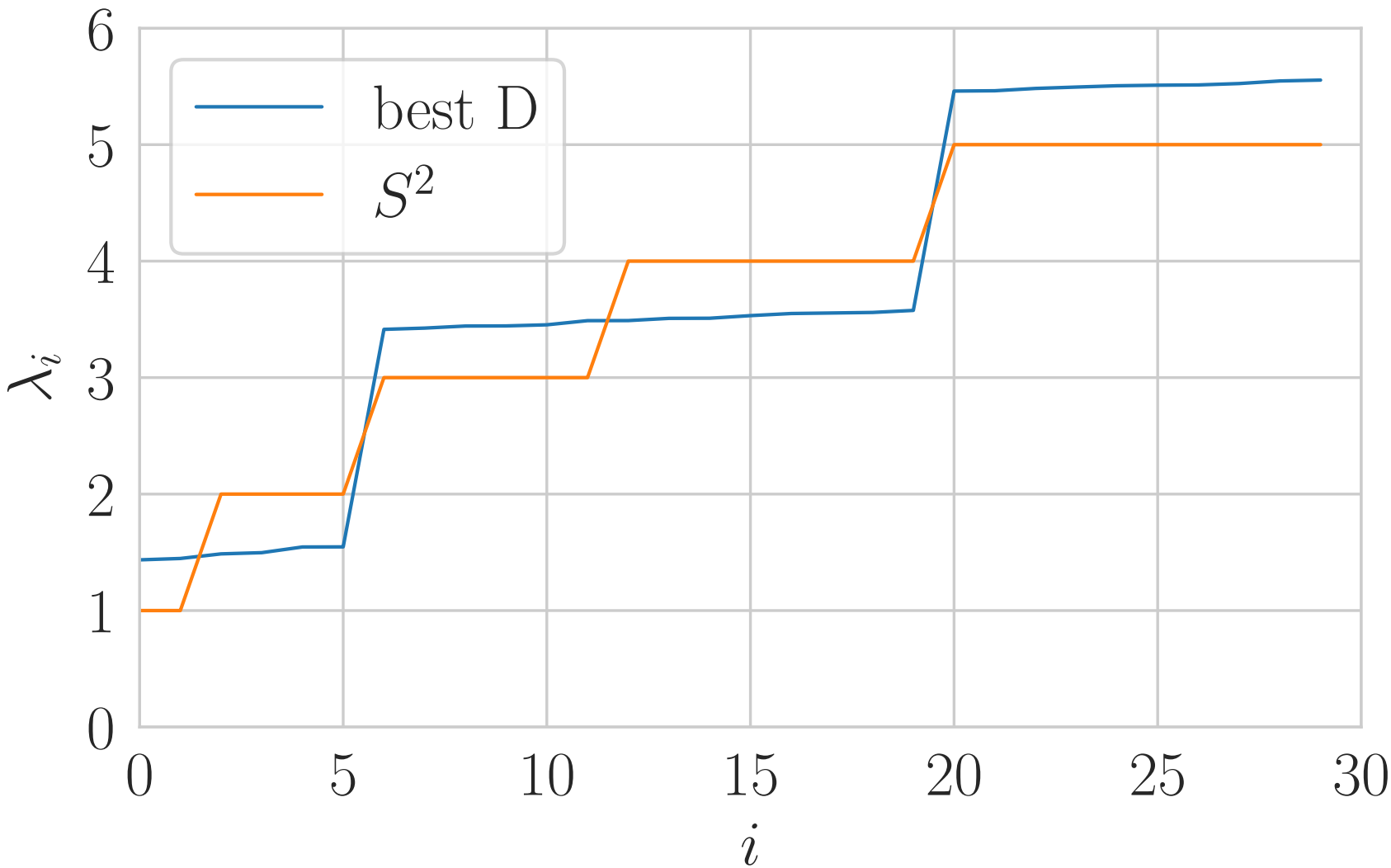
This is a (truncated) sphere



This is not a sphere!

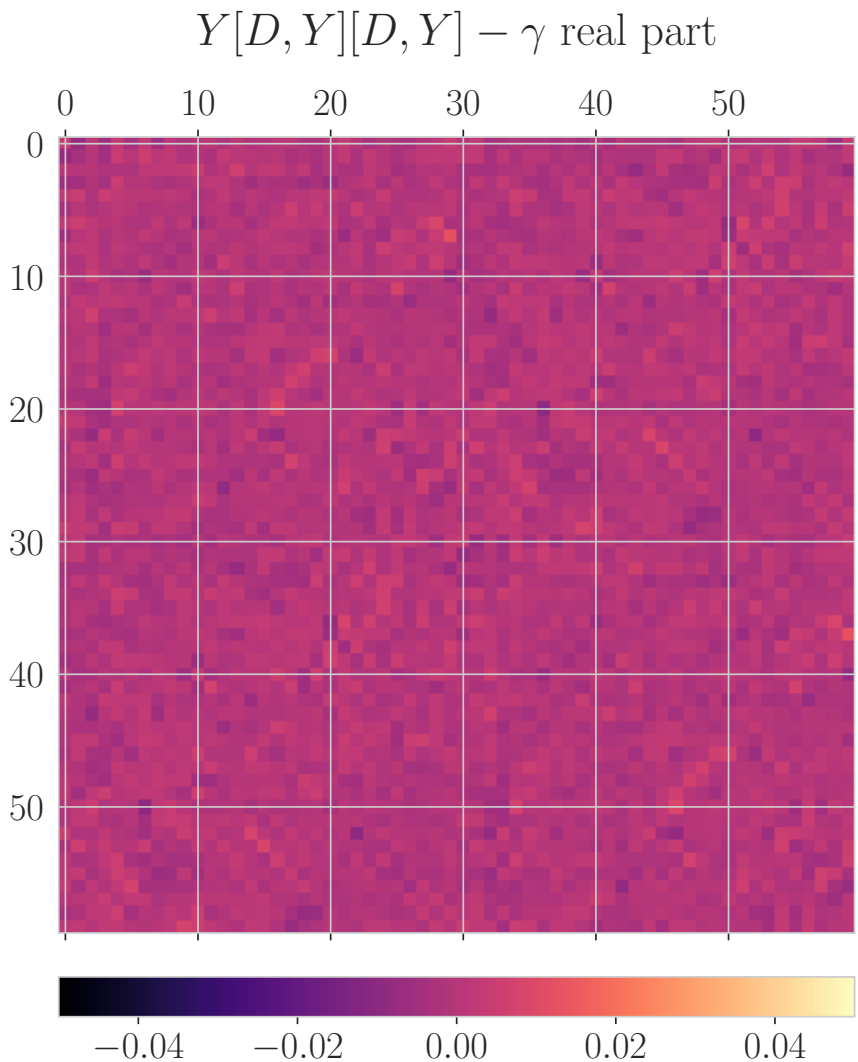
(but minimizes the Heisenberg equation constraint)

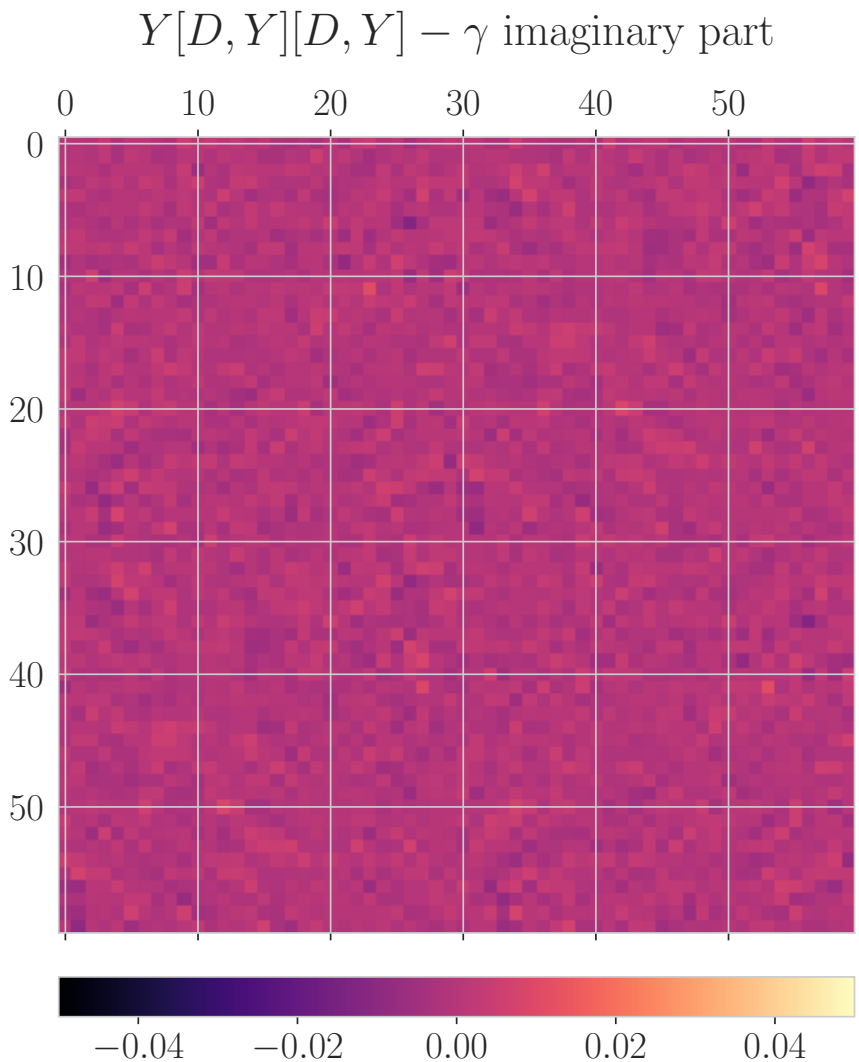
Positive part of the eigenvalue spectrum



This is not a sphere!

(but minimizes the Heisenberg equation constraint)



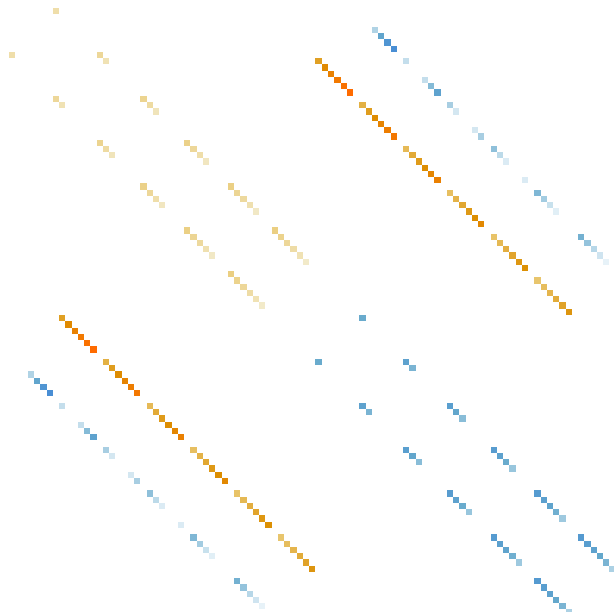


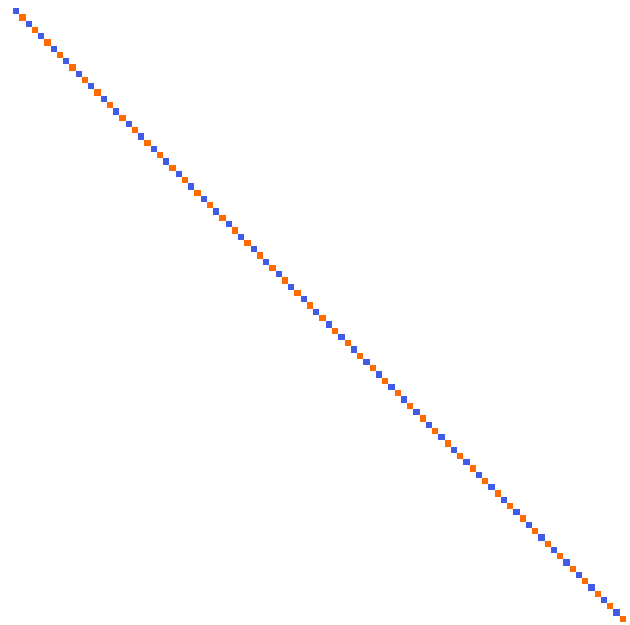
Analytic confirmation

Is the sphere the only minimum?

$$|| \langle Y[D, Y][D, Y] \rangle - \gamma ||_{HS}$$

- Maybe finite matrix size changes things?
- Y, γ vanish outside of some bands, in eigenbasis of D^{S^2}

$Y =$ 

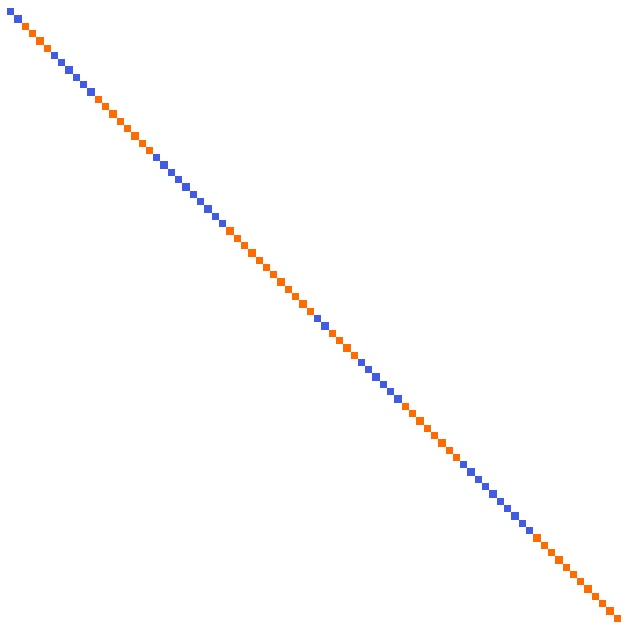
$\gamma =$ 

Analytic confirmation

Solution:

All operators of the form $D + cB$, where $B = sin(\pi D)$ and $c \in \mathbb{C}$ satisfy the Heisenberg relation in the infinite case.

This means B is a, bounded, operator with entries ± 1

$B =$ 

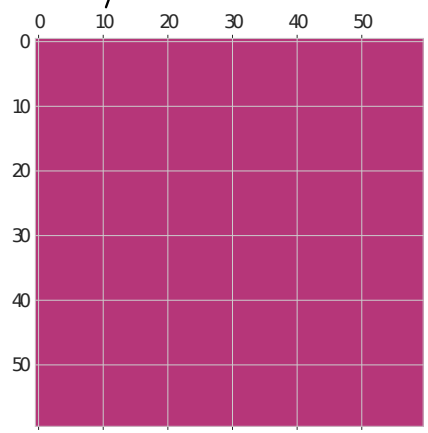
Analytic confirmation

Solution:

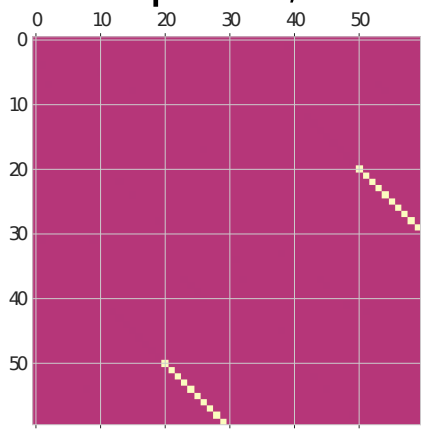
All operators of the form $D + cB$, where $B = sin(\pi D)$ and $c \in \mathbb{C}$ satisfy the Heisenberg relation in the infinite case.

When we truncate both D^{S^2} and the Heisenberg equation

$c = \pm 1/2$ solve exactly[a]



$c = 0$, the sphere, does not



[a] if truncation is odd/ even

Analytic confirmation

First order condition

The reason the solutions with $c \neq 0$ are not relevant at infinite size is that they do not satisfy the first order constraint

$$[[f(D^{S^2}), a\triangleright], \triangleleft b] = 0 \qquad a, b \in \mathcal{A},$$

and hence do not correspond to a spectral triple.
However the defect at finite size is similar for both cases.

Analytic confirmation

First order condition

The reason the solutions with $c \neq 0$ are not relevant at infinite size is that they do not satisfy the first order constraint

$$[[f(D^{S^2}), a\triangleright], \triangleleft b] = 0 \qquad a, b \in \mathcal{A},$$

and hence do not correspond to a spectral triple.
However the defect at finite size is similar for both cases.

- What do operator systems say to the first order condition?
- What is the difference between these geometries and the sphere?

Non-commutative distance

Distance measure in non-commutative geometry

(A. Connes, Noncommutative Geometry. (Academic Press, 1994))

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}} \{|\omega_1(a) - \omega_2(a)| : ||[D, a]|| \leq 1\}$$

Idea:

If we can calculate this numerically we can plot our geometry!
Maybe we can see a difference between the two Dirac operators?

Aim:

Find states of small dispersion, then use the distance between states of small dispersion to build a picture of M.

Non-commutative distance

Distance measure in non-commutative geometry

(A. Connes, Noncommutative Geometry. (Academic Press, 1994))

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}} \{|\omega_1(a) - \omega_2(a)| : ||[D, a]|| \leq 1\}$$

Questions:

- What are algebra elements a?
  Dirac we find has same eigenstates as sphere, can use truncated spherical harmonics as basis for $P_n C^\infty(S^2) P_n$
- Which states ω?
  Use coherent states, inspired by

  (L. Schneiderbauer, H. Steinacker 2016, J.Phys. A49 285301)

Implementing the distance calculation

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}} \{|\omega_1(a) - \omega_2(a)| : ||[D, a]|| \leq 1\}$$

Can parametrise algebra elements as $a = \sum_i c_i a_i$ for a_i a basis of the algebra and then minimize

$$\sum_i c_i \left(\langle v | a_i | v \rangle - \langle w | a_i | w \rangle \right)$$

over the c_i under the constraint that

$$|\sum_i c_i [D, a_i]| \leq 1$$

Here we chose a_i as truncations of the spherical harmonics as basis of $P_n C^\infty(S^2) P_n$

How do we define states?

Coherent states

We use the dispersion and the embedding maps Y_i from the Heisenberg relations

$$\eta(\omega_k) = \sum_i \langle\omega|Y_i^2|\omega\rangle - \langle\omega|Y_i|\omega\rangle^2 + \sum_{j<k} \frac{c}{d(\omega_j, \omega_k)}$$

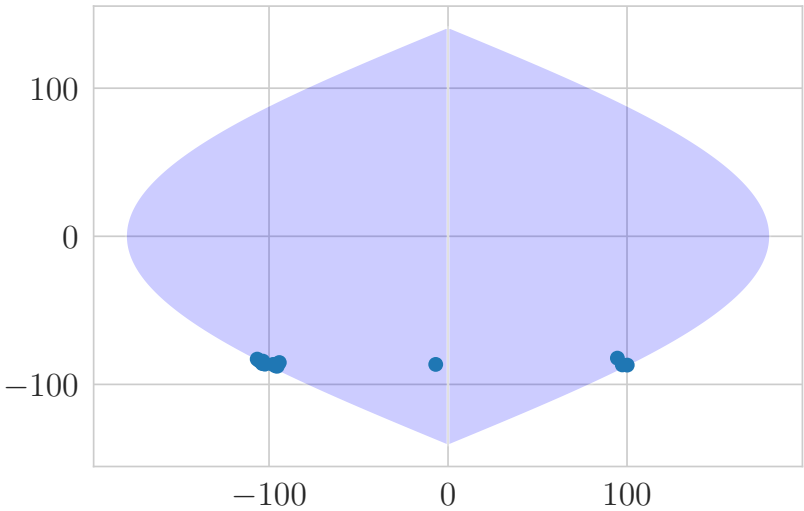
Now find a set of coherent states ω that minimizes this and plug them into distance equation. The repulsive potential is to ensure even distribution of points.

Shortcut!

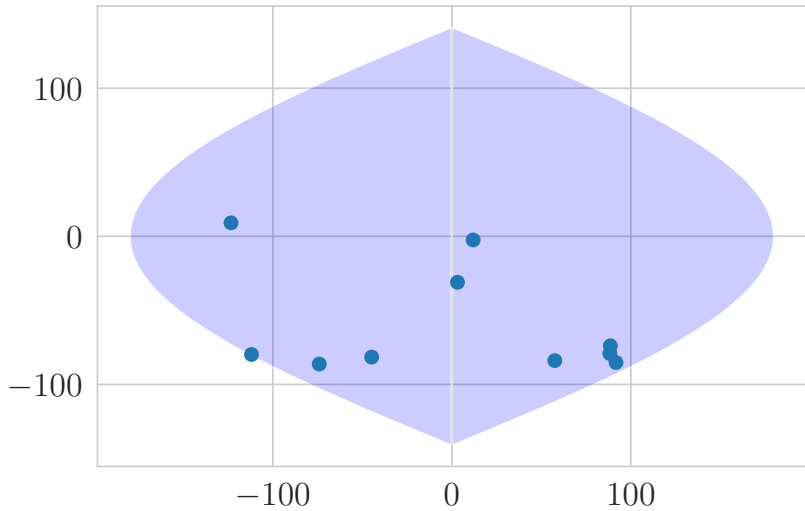
Strictly speaking we should optimize over all a, and all ω this is a non-convex double optimisation problem and not accessible, hence our use of Y_i as a proxy.

Effect of the repulsive potential

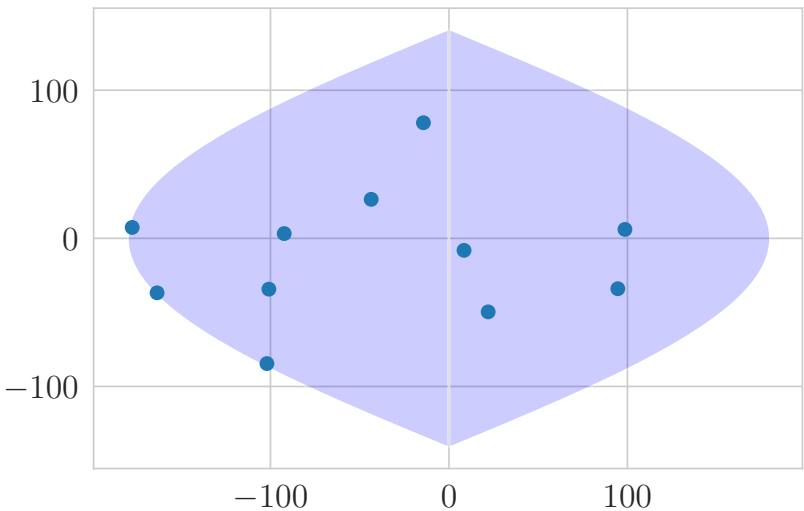
$c = 0$



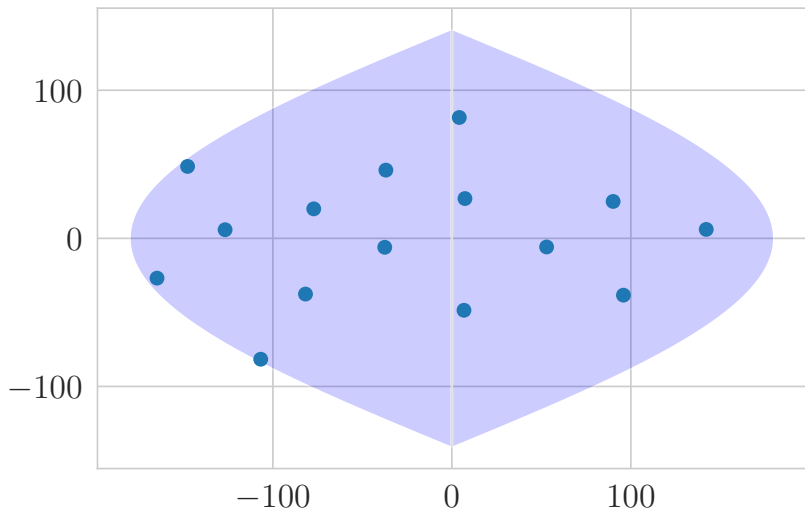
$c = 0.001$



$c = 0.1$



$c = 1000$



States are points

An element $\mathbf{v}$ of $\mathbb{P}(H_\Lambda)$ that is considered to be localized should be localized *somewhere*, that is, around some 'barycenter' $b(\mathbf{v}) \in M$.
We can prove:

Proposition

There exists a map $b\colon \mathbb{P}(H_\Lambda) \to M$ *such that*

$$|d_\Lambda(\mathbf{v},\mathbf{w}) - d_M(b(\mathbf{v}),b(\mathbf{w}))| = O(\sqrt{\eta(\mu_v)} + \sqrt{\eta(\mu_w)})$$

as $\eta(\mu_v), \eta(\mu_w) \to 0$, *uniformly in* $\mathbf{v}$, $\mathbf{w}$.

Points are states

The converse: Each point x in M can be approximated through a state $\mathbf{v}$ with small dispersion and with barycenter $b(\mathbf{v})$ close to x

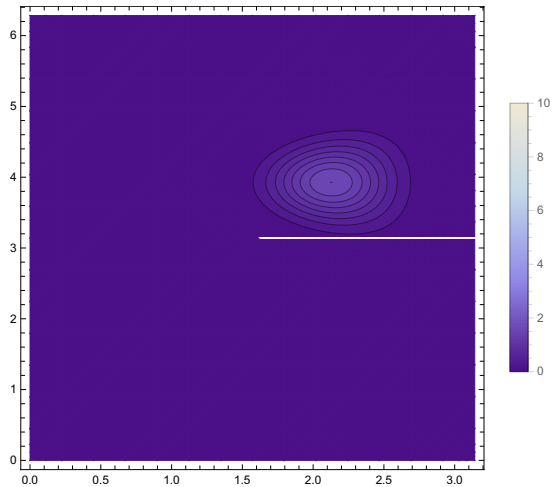
Proposition

Let M be equipped with a Dirac-type operator D on a Hermitian vector bundle $\pi\colon \mathbf{S} \to M$, and let $\widetilde{\pi}\colon \mathbb{P}(\mathbf{S}) \to M$ be its projectivized bundle. Then, there exists a family $\{\Phi_\Lambda\}_\Lambda$ of maps $\Phi_\Lambda : \mathbb{P}(\mathbf{S}) \to \mathbb{P}(H_\Lambda)$ such that for all $\epsilon > 0$,

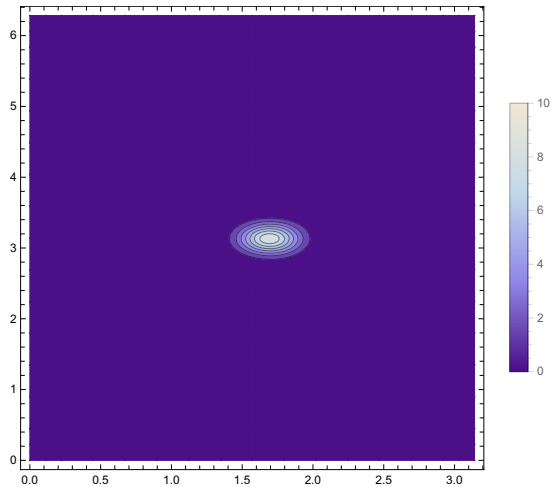
- $d_\Lambda(\Phi_\Lambda(v), \Phi_\Lambda(w)) = d_M(\widetilde{\pi}(v), \widetilde{\pi}(w)) + \widetilde{O}\left(\Lambda^{-1}\right)$ *uniformly.*
- *The dispersion $\eta(\mu)$ of the measure μ associated to $\Phi_\Lambda(v)$ is $\widetilde{O}\left(\Lambda^{-2}\right)$ uniformly.*
- *The maps Φ_Λ asymptotically invert b, in the sense that $d_M(\widetilde{\pi}(v), b(\Phi_\Lambda(v))) = \widetilde{O}\left(\Lambda^{-1}\right)$ uniformly and $d_\Lambda(\Phi_\Lambda(v)), \mathbf{v}) = \widetilde{O}\left(\sqrt{\eta(\mu_v)} + \Lambda^{-2}\right)$ uniformly whenever $b(\mathbf{v}) = \widetilde{\pi}(v)$.*

How does the dispersion change with Λ?

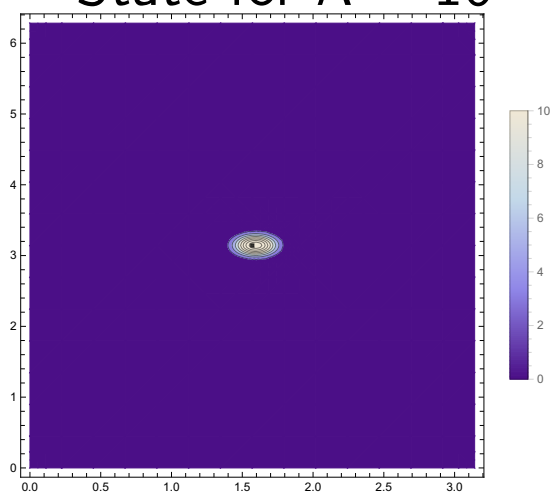
State for $\Lambda = 4$



State for $\Lambda = 10$



State for $\Lambda = 16$

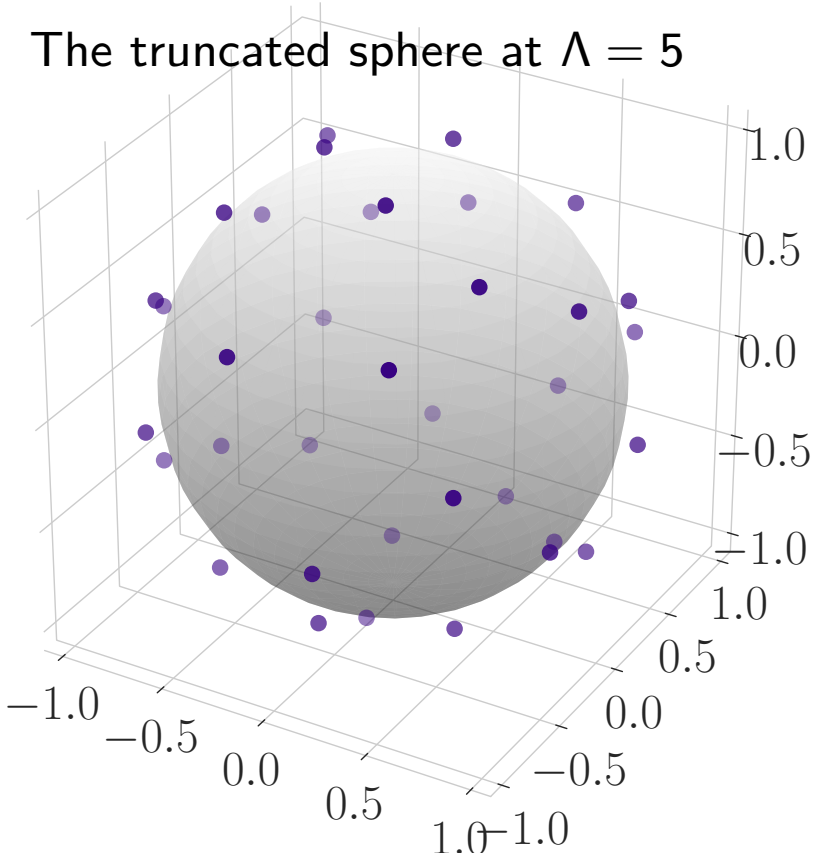


The algorithm for state generation

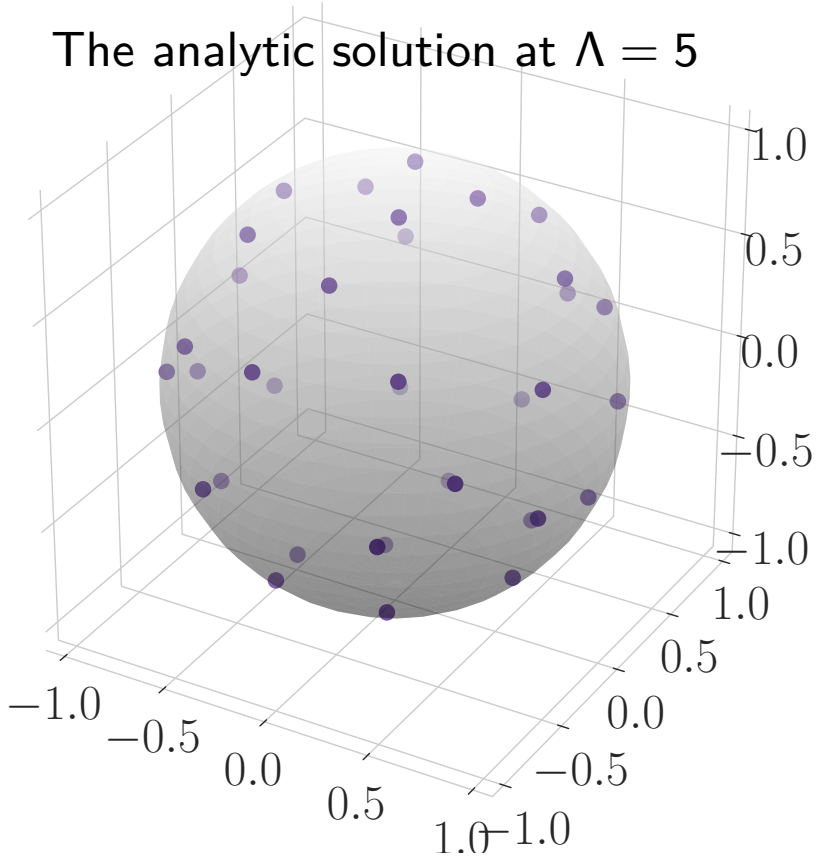
1: Find a vector v_0 (globally) minimizing η. Set $V = \{v_0\}$.
2: **while** $\sqrt{\eta(v)} + \sqrt{\eta(w)} \leq \alpha d(v, w)$ for $v \neq w \in V$, **do**
3: Find a vector w (locally) minimizing $e(w; V)$.
4: Append w to V.
5: **for** $v \in V$, **do**
6: Set $d(v, w) = \min\{|\langle v, av\rangle - \langle w, aw\rangle| : |[D, a]| \leq 1\}$.
7: **end for**
8: **end while**

A picture of geometry

The truncated sphere at $\Lambda = 5$



The analytic solution at $\Lambda = 5$



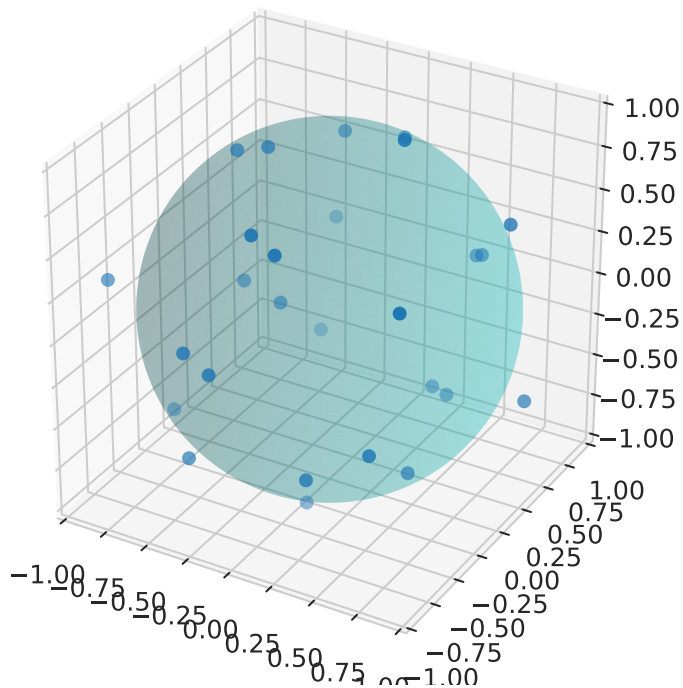
- run the algorithm to generate a set of states and their distance matrix
- use graph embedding algorithm to find a locally isometric embedding
- wonder why the analytic solution is smaller

Ongoing work

The algorithm implements algebra elements as $a = \sum_i c_i a_i$ linear combinations of spherical harmonics as a basis of $P_n C^\infty(S^2) P_n$

- Could we take linear combinations of a most general basis of algebra elements instead?

Looks promising:



Summary

Todays story:

- Exploring NCG using computer simulations
- truncated NCGs as basis for simulations
- first numerical tests of one sided Heisenberg relation and Connes distance function

Future plans:

- What is the difference between the two geometries?
- More visualisations:
  - try using wider range of a
  - use algorithm on other spectral triples
- spectral triples as matter in other QG approaches?

Thanks for tuning in!

Contact lisa.glaser@univie.ac.at or Twitter:@GravityWithHat

Todays story:

- Exploring NCG using computer simulations
- truncated NCGs as basis for simulations
- first numerical tests of one sided Heisenberg relation and Connes distance function

Future plans:

- What is the difference between the two geometries?
- More visualisations:
  - try using wider range of a
  - use algorithm on other spectral triples
- spectral triples as matter in other QG approaches?