Turaev-Viro state sum models with defects

NCG Seminar

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Work in progress with John Barrett

Motivation

Turaev-Viro Barrett-Westbury invariants

⇝ algebraic data: spherical fusion category

- closed oriented triangulated 3-mfold M $\longrightarrow$ number $Z(M) \in \mathbb{C}$
- oriented cpt triangulated 3-mfold M with $\partial M \neq \emptyset$ $\longrightarrow$ linear map $Z(M): Z(\partial M) \to \mathbb{C}$

⇝ topological invariant: triangulation independent

⇝ defines TQFT

⇝ physics applications:

- condensed matter physics: Kitaev models, Levin-Wen models
- quantum geometry: spin foam model, 3d quantum gravity

Why defects?

defects ~ distinguished submanifolds labeled with higher algebraic data

Turaev-Viro invariants with defects:

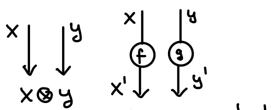
⇝
- explicit and simple model for defect TQFT
  [Carqueville-Runkel-Schaumann, ...]
- defects in condensed matter physics models
  [Kitaev-Kong, Bombin-Martin-Delgado, ...]
- observables in 3d quantum geometry
  [Barrett-Garcia-Islas-Martins, ...]

① Turaev-Viro Barrett-Westbury invariants

triangulated oriented 3-mfold $\mathcal{M}$
spherical fusion category $\mathcal{C}$ $\Big\}\longrightarrow$ mfold invariant $Z(M) \in \mathbb{C}$

spherical fusion category $\mathcal{C}$

- abelian, $\mathbb{C}$-linear
- monoidal
  - $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
  - tensor unit e
  - associator $a : \otimes(\otimes \times id) \xrightarrow{\sim} \otimes(id \times \otimes)$



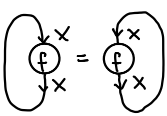
$f \otimes g : x \otimes y \to x' \otimes y'$

- pivotal
  duals and $** \cong id$
  - 

  - 

$ev_X : X^* \otimes X \to e$ $\quad$ $coev_X : e \to X \otimes X^*$

- spherical



$tr^L(f) = tr^R(f)$

- finite semisimple
  finite set I of simple objects
  every object direct sum of objects in I

Ex: $\operatorname{Rep} G^{fd}$, G finite group

- direct sums of reps
-
  - tensor product of reps
  - trivial rep on $\mathbb{C}$
  - rebracketing of tensor products
- reps on dual vector space V^*
- $V^{**} \cong V$ for $\dim V < \infty$

$ev_V : V^* \otimes V \to \mathbb{C}, \quad \alpha \otimes v \mapsto \alpha(v)$
$coev_V : \mathbb{C} \to V \otimes V^*, \quad z \mapsto z \sum_{i=1}^{n} v_i \otimes \alpha^i$

- trace of endomorphism $f : V \to V$

- finite set I of irreps
- reps $\cong$ direct sums of irreps

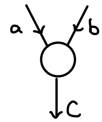
State Sum Construction

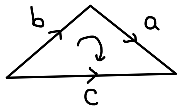
triangulated 3-mfold M $\rightsquigarrow$ labeling with algebraic data

- oriented edge e $\longrightarrow$ simple object $\ell(e) \in I$

  reversed edge — dual $\ell(e)^*$

- oriented triangle Δ $\longrightarrow$ morphism $\ell(\Delta): a \otimes b \to c$

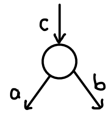




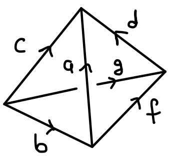
Opposite orientation

morphism $\ell(\Delta)': c \to a \otimes b$ via

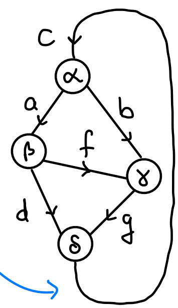
tr: $\mathrm{Hom}_{\mathcal{C}}(a \otimes b, c) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(c, a \otimes b)$



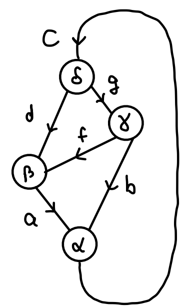
- oriented tetrahedron t $\longrightarrow$ 6j-Symbol $6j(t, \ell)$



$\alpha: c \to a \otimes b$
$\beta: a \to d \otimes f$
$\gamma: f \otimes b \to g$
$\delta: d \otimes g \to c$



trace in $\mathcal{C}$



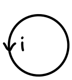
Opposite Orientation

$$\mathrm{Hom}(c, a \otimes b) \otimes \mathrm{Hom}(a, d \otimes f) \otimes \mathrm{Hom}(f \otimes b, g) \otimes \mathrm{Hom}(d \otimes g, c) \to \mathbb{C}$$

- associator $a: \otimes(\otimes \times id) \xrightarrow{\sim} \otimes(id \times \otimes)$ on simple objects
- independent of choices

State Sum

$$\boxed{Z(M, T) = \frac{1}{\dim \mathcal{C}^V} \sum_{\ell: E \to I} \left(\prod_{t \in T} 6j(t, \ell) \right) \left(\prod_{e \in E} \dim(\ell(e)) \right)}$$

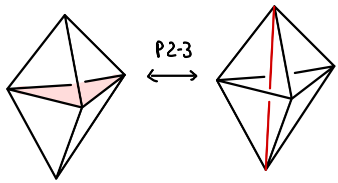
$\dim(i) = tr(1_i) =$  quantum dimension

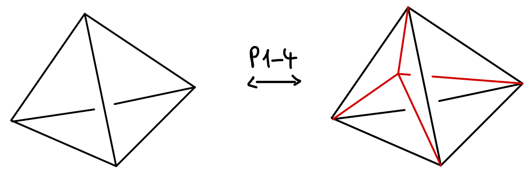
$\dim(\mathcal{C}) = \sum_{i \in I} \dim(i)^2$ dimension of $\mathcal{C}$

topological invariance

$$\mathcal{Z}(M,T) = \mathcal{Z}(M)$$

- triangulations of M related by finite sequences of Pachner moves





- invariance P2-3: Biedenharn-Elliott relations for 6js
  ~ pentagon for associator of $\mathcal{C}$

$$\begin{tikzcd}
((a\otimes b)\otimes c)\otimes d \arrow[r] \arrow[d] & (a\otimes b)\otimes(c\otimes d) \arrow[r] & a\otimes(b\otimes(c\otimes d)) \\
(a\otimes(b\otimes c))\otimes d \arrow[r] & a\otimes((b\otimes c)\otimes d) \arrow[ur] &
\end{tikzcd}$$

- invariance P1-4: orthogonality relations for 6js
  ~ invertibility of associator of $\mathcal{C}$

⇝ 3-mfold invariant: triangulation independence

⇝ extends to mfolds with boundary

⇝ defines TQFT

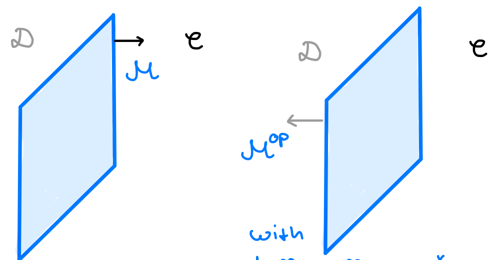
② Turaev-Viro state sums with defects [J.B., C.M.]

2.1. defect data

3d regions — spherical fusion categories

defect planes — finite semisimple bimodule categories with bimodule traces

- category $\mathcal{M}$
- action functors
  $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$
  $\triangleleft : \mathcal{M} \times \mathcal{D} \to \mathcal{M}$



with
$d \triangleright^{op} m \triangleleft^{op} c = c^* \triangleright m \triangleleft d^*$

- natural isomorphisms

$$\left.\begin{aligned} c &: \triangleright(\otimes \times id) \xrightarrow{\sim} \triangleright(id \times \triangleright) \\ d &: \triangleleft(\triangleleft \times id) \xrightarrow{\sim} \triangleleft(id \times \otimes) \\ b &: \triangleleft(\triangleright \times id) \xrightarrow{\sim} \triangleright(id \times \triangleleft) \\ \gamma &: e \triangleright - \xrightarrow{\sim} id_{\mathcal{M}} \\ \delta &: - \triangleleft e \xrightarrow{\sim} id_{\mathcal{M}} \end{aligned}\right\} \text{pentagon + triangle axioms}$$

- trace: morphisms $\Theta_m : End_{\mathcal{M}}(m) \to \mathbb{C}$ } cyclic, non-degenerate, compatible with actions
- finite set I of simple objects
  every object direct sum of simple objects

Ex spherical fusion category $\mathcal{C}$ as $(\mathcal{C},\mathcal{C})$-bimodule category
$\triangleright = \triangleleft = \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
$\Theta = tr_{\mathcal{C}}$

$$\begin{tikzcd} & U \arrow[dl, "f_1"'] \arrow[dr, "f_2", no head] & \\ G_1 & & G_2 \end{tikzcd}$$

Ex: U, G_1, G_2 finite groups, group homomorphisms
$\mathrm{Rep}\, U^{fd}$ is $(\mathrm{Rep}\, G_1^{fd}, \mathrm{Rep}\, G_2^{fd})$ - bimodule category

$\triangleright : \mathrm{Rep}\, G_1^{fd} \times \mathrm{Rep}\, U^{fd} \to \mathrm{Rep}\, U^{fd}$, $(V,W) \mapsto V \otimes W$ ⟵ with induced representation of U
$\triangleleft : \mathrm{Rep}\, U^{fd} \times \mathrm{Rep}\, G_2^{fd} \to \mathrm{Rep}\, U^{fd}$, $(W,X) \mapsto W \otimes X$

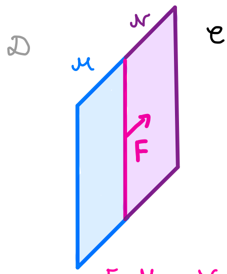
$\Theta = tr_{Vect}$

defect lines – bimodule functors preserving traces

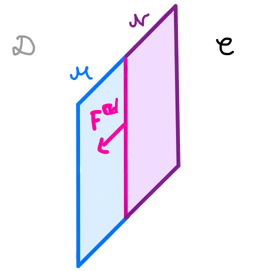
- functor $F: \mathcal{M} \to \mathcal{N}$
- natural isomorphisms

$$\left.\begin{aligned} s&: F\triangleright \xrightarrow{\sim} \triangleright(\mathrm{id}\times F) \\ t&: \triangleleft(F\times \mathrm{id}) \xrightarrow{\sim} F\triangleleft \end{aligned}\right\} \text{pentagon + triangle + hexagon axioms}$$

- compatibility condition with trace



$F: \mathcal{M} \to \mathcal{N}$



adjoint $F: \mathcal{N} \to \mathcal{M}$

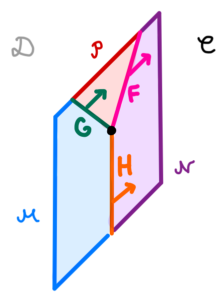
Ex
- $F = c\otimes - : \mathcal{C} \to \mathcal{C}$ with $c \in \mathcal{Z}(\mathcal{C})$
- $F = c\triangleright - : \mathcal{M} \to \mathcal{M}$ $\quad c \in \mathcal{Z}(\mathcal{C})$
- $F = -\triangleleft d : \mathcal{M} \to \mathcal{M}$ $\quad d \in \mathcal{Z}(\mathcal{D})$
- group homomorphisms

$$\begin{tikzcd} & U \arrow[dl, "f_1"'] \arrow[dr, "f_2"] \arrow[dd, "h"] & \\ G_1 & & G_2 \\ & V \arrow[ul, "g_1"] \arrow[ur, "g_2"'] & \end{tikzcd}$$

$\Rightarrow (\mathrm{Rep}\,G_1^{fd}, \mathrm{Rep}\,G_2^{fd})$-bimodule functor $F: \mathrm{Rep}\,V^{fd} \longrightarrow \mathrm{Rep}\,U^{fd}$

defect points – bimodule natural transformations

- natural transformations compatible with bimodule structure



$\nu: FG \to H$

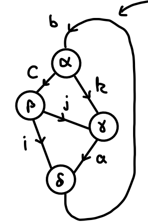
Ex
- $f \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(c, c') \Rightarrow \nu = f\triangleright - : c\triangleright - \to c'\triangleright -$
- conjugation of group homomorphisms

2.2. defect data $\longrightarrow$ generalised 6j symbols

$\rightsquigarrow$ from coherence morphisms of defect data on simple objects

- spherical fusion category $\mathcal{C}$:

  $\rightsquigarrow$ associator $\alpha: \otimes(\otimes \times id) \xrightarrow{\sim} \otimes(id \times \otimes)$



trace in $\mathcal{C}$

$\alpha: b \to c \otimes k$

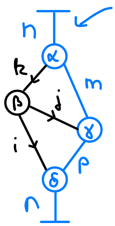
$\beta: c \to i \otimes j$

$\gamma: j \otimes k \to a$

$\delta: i \otimes a \to b$

- $(\mathcal{C},\mathcal{D})$-bimodule category $\mathcal{M}$:

  $\rightsquigarrow$ $c: \triangleright(\otimes \times id) \xrightarrow{\sim} \triangleright(id \times \triangleright)$ $\qquad$ $d: \triangleleft(\triangleleft \times id) \xrightarrow{\sim} \triangleleft(id \times \otimes)$



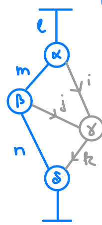
bimodule trace in $\mathcal{M}$

$\alpha: n \to k \triangleright m$

$\beta: k \to i \otimes j$

$\gamma: j \triangleright m \to p$

$\delta: i \triangleright p \to n$



bimodule trace in $\mathcal{M}$

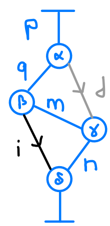
$\alpha: l \to m \triangleleft i$

$\beta: m \to n \triangleleft j$

$\gamma: j \otimes i \to k$

$\delta: n \triangleleft k \to l$

$b: \triangleleft(\triangleright \times id) \xrightarrow{\sim} \triangleright(id \times \triangleleft)$



$\alpha: p \to q \triangleleft j$

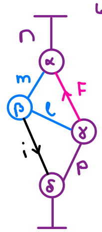
$\beta: q \to i \triangleright m$

$\gamma: m \triangleleft j \to n$

$\delta: i \triangleright n \to p$

- bimodule functor $F: \mathcal{M} \to \mathcal{N}$

  $\rightsquigarrow$ $s: F \triangleright \xrightarrow{\sim} \triangleright(id \times F)$ $\qquad$ $t: \triangleleft(F \times id) \xrightarrow{\sim} F \triangleleft$



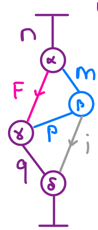
bimodule trace in $\mathcal{N}$

$\alpha: n \to F(m)$

$\beta: m \to i \triangleright l$

$\gamma: F(l) \to p$

$\delta: i \triangleright p \to n$



bimodule trace in $\mathcal{N}$

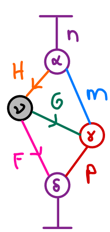
$\alpha: n \to F(m)$

$\beta: m \to p \triangleleft i$

$\gamma: F(p) \to q$

$\delta: q \triangleleft i \to n$

- bimodule natural transformation $\nu: H \to FG$



$\alpha: n \to H(m)$

$\nu_m: H(m) \to FG(m)$

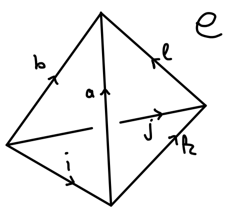
$\gamma: G(m) \to p$

$\delta: F(p) \to n$

2.3. State sum model with defects [J.B., C.M.]

- oriented triangulated 3d mfold M
- defect data in dual cell complex

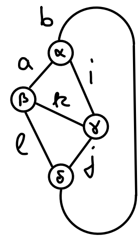
tetrahedra **6j symbols** $\in \mathbb{C}$

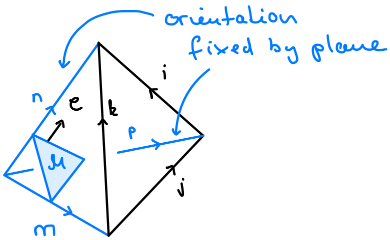


associator for $\mathcal{C}$

$a: \otimes(\otimes \times id) \xrightarrow{\sim} \otimes(id \times \otimes)$

Symmetries: S_4

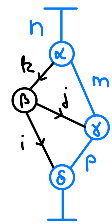


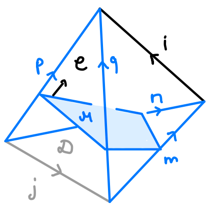


coherence data for action functor $\triangleright: \mathcal{C}\times\mathcal{M}\to\mathcal{M}$

$c: \triangleright(\otimes\times id) \xrightarrow{\sim} \triangleright(id\times\triangleright)$

Symmetries: S_3

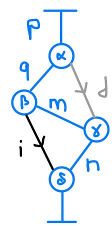


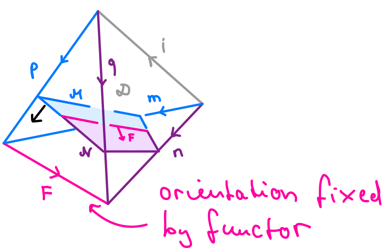


Coherence data for $\triangleright: \mathcal{C}\times\mathcal{M}\to\mathcal{M}$, $\triangleleft: \mathcal{M}\times\mathcal{D}\to\mathcal{M}$

$b: \triangleleft(\triangleright\times id) \xrightarrow{\sim} \triangleright(id\times\triangleleft)$

Symmetries: $\mathbb{Z}_2\times\mathbb{Z}_2$

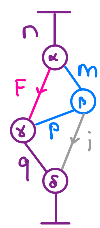


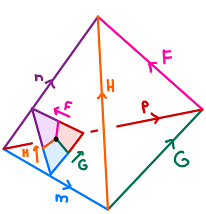


Coherence data for $\mathcal{D}$-module functor $F: \mathcal{M}\to\mathcal{N}$

$t: \triangleleft(F\times id) \xrightarrow{\sim} F\triangleleft$

Symmetries: $\mathbb{Z}_2$

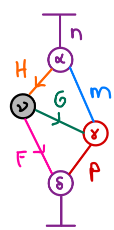




bimodule natural transformation

$\nu: H \Rightarrow FG$

Symmetries: id



State sum

$$\boxed{\mathcal{Z}(M,T) = \frac{1}{\prod_{\text{regions } r} \dim(e_r)^{V_r}} \sum_{\substack{\ell: e_r \to I_r \\ e_p \to I_p}} \left(\prod_{t\in T} 6j(t,\ell)\right)\left(\prod_{e\in E_r \cup E_p} \dim \ell(e)\right)}$$

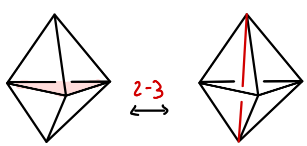
dimensions of spherical fusion categories for regions

labeling of edges with simple objects in spherical fusion categories and bimodule categories

- independent of choices (edge orientation of tetrahedra etc) $\rightsquigarrow$ properties of defect data
- 6j symbols encode coherence isomorphisms of spherical and defect data

topological invariance $\rightsquigarrow$ Pachner move invariance

$\rightsquigarrow$ direct generalisation:



2-3 move:

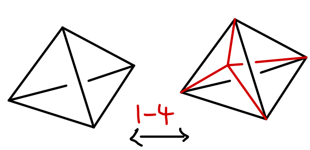
pentagon relation for coherence isos

$a: \otimes(\otimes \times id) \xrightarrow{\sim} \otimes(id \times \otimes)$ spherical categories

$$\left.\begin{aligned} &c: \triangleright(\otimes \times id) \xrightarrow{\sim} \triangleright(id \times \triangleright) \\ &d: \triangleleft(\triangleleft \times id) \xrightarrow{\sim} \triangleleft(id \times \otimes) \\ &b: \triangleleft(\triangleright \times id) \xrightarrow{\sim} \triangleright(id \times \triangleleft) \end{aligned}\right\} \text{ bimodule categories}$$

$$\left.\begin{aligned} &s: F\triangleright \xrightarrow{\sim} \triangleright(id \times F) \\ &t: \triangleleft(F \times id) \xrightarrow{\sim} F\triangleleft \end{aligned}\right\} \text{ bimodule functors}$$

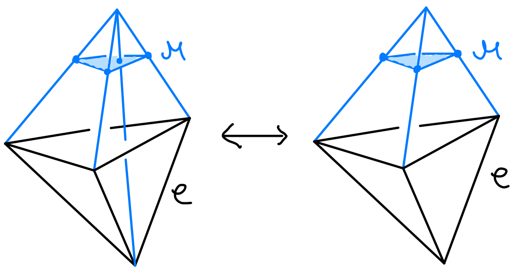
bimodule compatibility of bimodule natural transformation



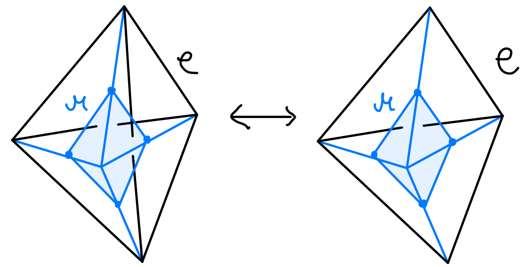
1-4 move:

invertibility of coherence isos

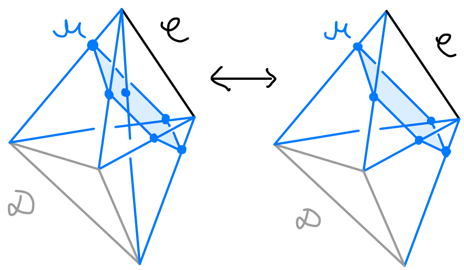
2-3 moves



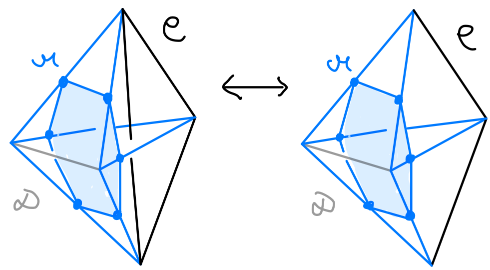
𝒞 - module category



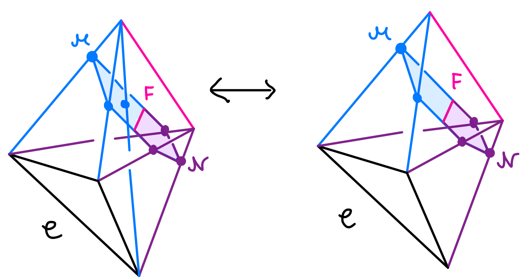
$c: \triangleright(\otimes \times \mathrm{id}) \xrightarrow{\sim} \triangleright(\mathrm{id} \times \triangleright)$



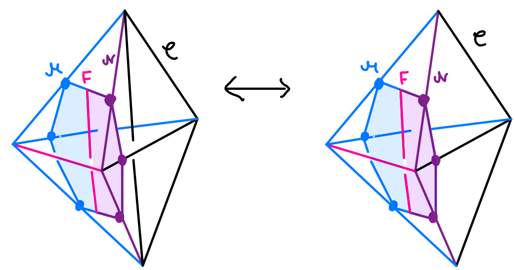
$(\mathcal{C}, \mathcal{D})$ - bimodule category



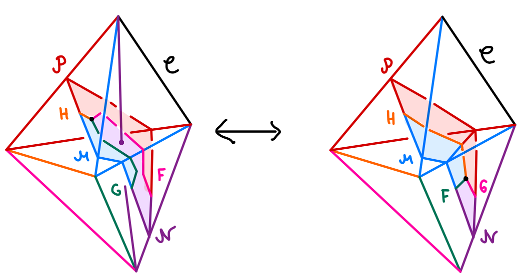
$b: \triangleleft(\triangleright \times \mathrm{id}) \xrightarrow{\sim} \triangleright(\mathrm{id} \times \triangleleft)$



𝒞 - module functor

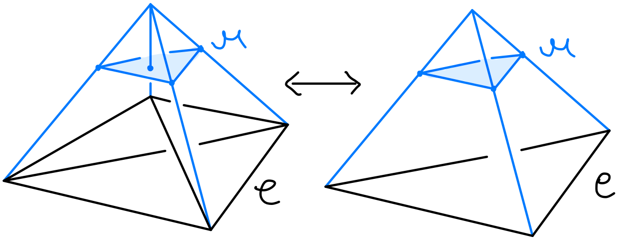


$s: \triangleright(\mathrm{id} \times F) \xrightarrow{\sim} F \triangleright$

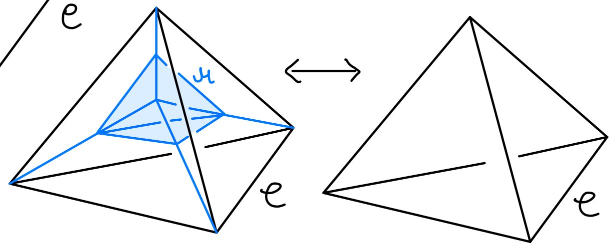


𝒞 - module natural transformation

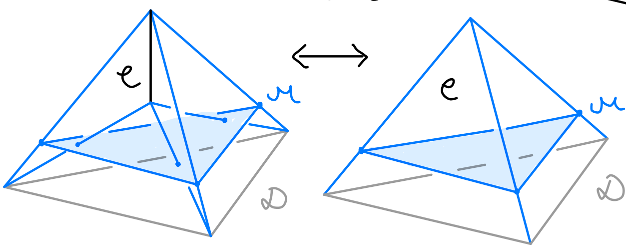
1-4 moves

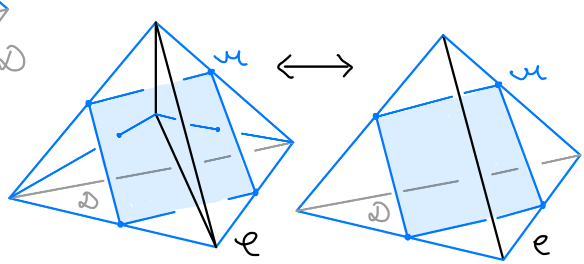


$\mathcal{C}$-module category

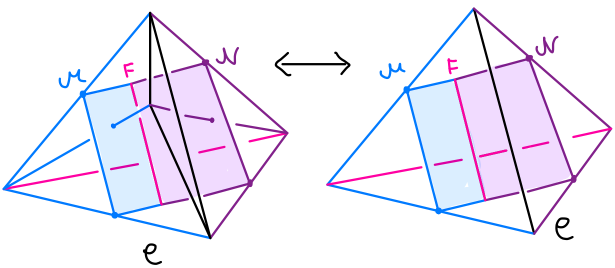


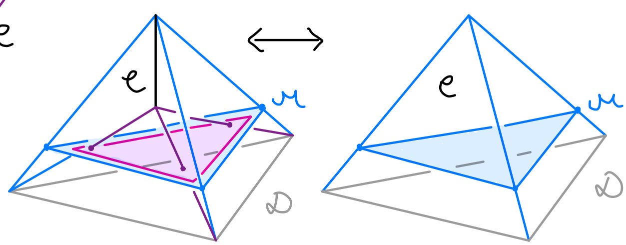
$(\mathcal{C},\mathcal{D})$-bimodule category



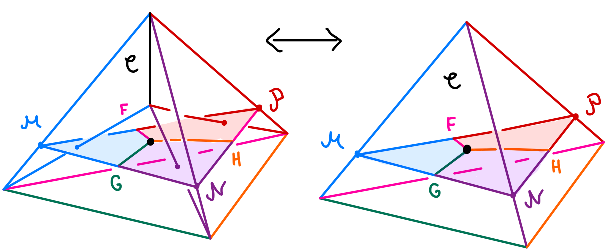


$\mathcal{C}$-module functor





$\mathcal{C}$-module natural transformation



$\Rightarrow$ triangulation independence

Summary:

- Simple and explicit state sum model with defects
- tetrahedra ⇝ generalised 6j symbols
  ⇝ coherence isomorphisms of spherical data and defect data on simple objects
- Pachner invariance
  2-3 move ⇝ pentagon relations for coherence isomorphisms
  1-4 move ⇝ invertibility of coherence isomorphisms

⇝ direct generalisation of Turaev-Viro invariants

To Do

- examples:
- $\mathcal{M} = \mathcal{N} = \dots = \mathcal{C}$ as $(\mathcal{C},\mathcal{C})$-bimodule category
  bimodule functors - objects of $\mathcal{Z}(\mathcal{C})$
  bimodule nat. transformations - morphisms of $\mathcal{Z}(\mathcal{C})$
  ⇝ „excitations" in condensed matter physics models
- simple examples related to group representations

- extend to case $\partial M \neq \emptyset$
⇝ related to defect TQFT

Thank you for your attention!