Cyclic cocycles for proper Lie group actions

Xiang Tang

Washington University in St. Louis

February 23, 2021

Outline

In this talk, we will report our exploration of cyclic cohomology for proper Lie group actions. We will introduce explicit cyclic cocycles on the Harish-Chandra Schwartz algebra using the geometry of Lie groups. As applications, we will present index theorems for proper cocompact Lie group actions.

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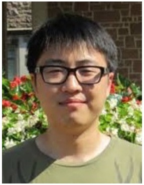
1. Invariant elliptic operators
2. Cyclic cocycles for proper actions
3. Pairing with K-theory

My Collaborators





This talk is based on joint work with Pierre Clare, Nigel Higson, Peter Hochs, Markus Pflaum, Hessel Posthuma, and Yanli Song.

Index

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Let V_μ be an irreducible representation of K with highest weight μ. On the associated vector bundle $\widetilde{V}_\mu := G \times_K V_\mu$, we consider the operator

$$\not{D}_\mu : \Gamma(X, \mathcal{S}^+ \otimes \widetilde{V}_\mu) \to \Gamma(X, \mathcal{S}^- \otimes \widetilde{V}_\mu).$$

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Theorem (Atiyah-Schmid)

If G has Harish-Chandra's discrete series representations, the (co)kernel of $\not{D}_\mu$ is a discrete series representation of G.

The Connes-Kasparov isomorphism

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The index of the operator $\not{D}_\mu$ is the element

$$\mathrm{Ind}(\not{D}_\mu) := [\ker(\not{D}_\mu)] - [\mathrm{coker}(\not{D}_\mu)] \in K_0(C_r^*(G)).$$

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Conjecture (Connes-Kasparov)

The index morphism

$$\mathrm{Ind} : \mathfrak{Rep}(K) \longrightarrow K_0(C_r^*(G))$$

is an isomorphism of abelian groups.

The Connes-Kasparov conjecture

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Theorem (Chabert-Echterhoff-Nest)

Let G be a second countable almost connected group (i.e. G/G_0 is compact, where G_0 denotes the connected component of G). Then the Baum-Connes assembly map

$$\mathrm{Ind} : K_\bullet^{top}(G) \to K_\bullet\big(C_r^*(G)\big)$$

is an isomorphism.

L^2-index theorem

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Theorem (Connes-Moscovici)

Assume that G is unimodular. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K.

$$\mathrm{tr}(\mathrm{Ind}(\not{D}_\mu)) = \langle \widehat{A}(\mathfrak{g}, K) \wedge \mathrm{ch}(V_\mu)_{\mathfrak{m}^*}, [V] \rangle,$$

where $\mathfrak{m}^* \subset \mathfrak{g}^*$ is the conormal space of $\mathfrak{k}$ in $\mathfrak{g}$, and $[V]$ is the fundamental class of $\mathfrak{m}^*$.

The main questions

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Use the geometry of G to construct explicit cyclic cocycles on $\mathcal{C}(G) \subset C_r^*(G)$ generalizing the L^2-trace on $C_r^*(G)$.

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Use the geometry of G to construct explicit cyclic cocycles on $\mathcal{C}(G) \subset C_r^*(G)$ generalizing the L^2-trace on $C_r^*(G)$.

Question

Compute the topological formula for the index pairing between the cyclic cocycles and $K_0(C_r^*(G))$.

Differential currents on $\widehat{\mathbb{R}}^n$

When $G = \mathbb{R}^n$, $\big(\mathcal{C}(\mathbb{R}^n), *\big) = \big(\mathcal{S}(\widehat{\mathbb{R}}^n), \cdot\big)$.

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$$H_\bullet(\widehat{R}^n) = \begin{cases} \mathbb{R}, & \bullet = n, \\ 0, & \text{otherwise.} \end{cases}$$

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On $\mathcal{S}(\widehat{\mathbb{R}}^n)$, $H_n(\widehat{R}^n)$ is generated by a degree n differential current,

$$\Psi(f_0, \cdots, f_n) = \int_{\mathbb{R}^n} f_0 df_1 \cdots df_n.$$

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Define a function $C : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} \to \mathbb{R}$ by

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Define Φ to be a cocycle on $\mathcal{C}(\mathbb{R}^n)$ by

$$\Phi(f_0, \cdots, f_n) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} dx_1 \cdots dx_n C(x_1, \cdots, x_n) \\ f_0(-x_1 - \cdots - x_n) f_1(x_1) \cdots f_n(x_n).$$

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$$\begin{aligned} & \delta(\varphi)(g_1, \cdots, g_{k+1}) \\ = \ & \varphi(g_2, \cdots, g_k) \\ - \ & \varphi(g_1 g_2, \cdots, g_{k+1}) + \cdots + (-1)^k \varphi(g_1, \cdots, g_k g_{k+1}) \\ + \ & (-1)^{k+1} \varphi(g_1, \cdots, g_k). \end{aligned}$$

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\end{aligned}$$

The differentiable group cohomology $H^\bullet_{\mathrm{diff}}(G)$ is defined to be the cohomology of $(C^\infty(G^{\times\bullet}),\delta)$.

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\begin{aligned}
< \hat{\varphi}, f_0 \otimes \cdots \otimes f_k > \quad := & \int f_0(g_k^{-1} \cdots g_1^{-1}) f_1(g_1) \cdots f_k(g_k) \\
& \varphi(g_1, \cdots, g_k) dg_1 \cdots dg_k
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Theorem (Pflaum-Posthuma-T, Piazza-Posthuma)

The above pairing descends to a character morphism χ,

$$\chi : H^\bullet_{\mathrm{diff}}(G) \to HP^\bullet(\mathcal{C}(G)).$$

Example of $SL_2(\mathbb{R})$

Let $SL(2, \mathbb{R})$ be the Lie group of 2×2 real matrices with determinant being 1, e.g.

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | ad - bc = 1 \right\}.$$

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The function A is a 2-cocycle on $SL_2(\mathbb{R})$, and $\chi(A)$ is the Chern character of the fundamental Fredholm module of Alain Connes.

Orbital Integrals

For $x \in G$, let $Z_G(x)$ be the centralizer of x in G and $d_{G/Z_G(x)}\dot{g}$ be the left invariant measure on $G/Z_G(x)$.

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is an important tool in representation theory with deep connections to number theory.
An important property is that for regular $x \in H$, a Cartan subgroup of G, the orbital integral defines a trace tr_x on $\mathcal{C}(G)$, i.e.

$$\mathrm{tr}_x(f) := \Lambda_f^{Z_G(x)}.$$

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Let $P = MAN$ be a cuspidal parabolic subgroup of G. Using the decomposition $G = KMAN$, we introduce a generalization of the determinant function

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$$C : C^\infty(K \times G^{\times m}),$$

for $m = \dim(A)$.
For a semisimple element $x \in M$, define a degree m cocycle on $\mathcal{C}(G)$ by

$$\Phi_{P,x}(f_0, f_1, \ldots, f_m) := \int_{h \in M/Z_M(x)} \int_{KN} \int_{G^{\times m}} dhdkdndg_1 \cdots dg_m \\ C(k, g_1 g_2 \ldots g_m, \ldots, g_{m-1} g_m, g_m) f_0\big(khxh^{-1}nk^{-1}(g_1 \ldots g_m)^{-1}\big) \\ f_1(g_1) \ldots f_m(g_m).$$

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- $\partial\Phi_{P,x} = 0$, *e.g.*

$$\Phi_{P,x}(f_0 * f_1, f_2, \cdots, f_{m+1}) - \Phi_{P,x}(f_0, f_1 * f_2, \cdots, f_{m+1}) + \cdots + (-1)^{m+1}\Phi_{P,x}(f_{m+1} * f_0, \cdots, f_m) = 0.$$

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- $\Phi_{P,x}$ *is cyclic, e.g.*

$$\Phi_{P,x}(f_m, f_0, \cdots, f_{m-1}) = (-1)^m \Phi_{P,x}(f_0, \cdots, f_m).$$

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Question

Compute the pairing between the above cocycles and $K_{\bullet}(\mathcal{C}(G)) \cong K_{\bullet}(C^*_r(G))$.

Higher L^2-index theorem

Theorem (Pflaum-Posthuma-T)

Let G be a Lie group acting properly and cocompactly on a manifold X. Suppose that D is an elliptic G-invariant differential operator on X, and $[\varphi] \in H^{2k}_{\text{diff}}(G; L)$.

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$$\chi(\varphi)(\mathrm{Ind}(D)) = \frac{1}{(2\pi\sqrt{-1})^k(2k)!}\int_{T^*X} c\Phi([\varphi]) \wedge \hat{A}(T^*X) \wedge \mathrm{ch}(\sigma(D)),$$

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where $c \in C^\infty_{\text{cpt}}(X)$ is a cut-off function, and Φ is the characteristic class map from $H^\bullet_{\text{diff}}(G; L)$ to the de Rham cohomology of G-invariant differential forms on X.

L^2-index theorem for proper cocompact actions

When G is unimodular, the previous index formula for $\varphi = 1 \in H^0_{\mathrm{diff}}(G)$ gives Hang Wang's L^2-index theorem for G-invariant elliptic operators on a manifold with a proper and cocompact action.

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The previous theorem holds true for proper cocompact Lie groupoid actions. For example, let $\mathsf{H}_{\mathcal{F}}$ be the holonomy groupoid of a regular foliation $\mathcal{F}$ on M. Assume that $\mathsf{H}_{\mathcal{F}}$ is unimodular. The index formula for $[\varphi] = 1 \in H^0_{\mathrm{diff}}(\mathsf{H}_{\mathcal{F}})$ gives the Connes index theorem for measured foliations.

Higher orbital integrals and fixed point theorem

- X is equipped with a G-equivariant Spin^c-structure,
- $S \to X$ is the corresponding spinor bundle,
- $W \to X$ is a G-equivariant Hermitian vector bundle,
- $D\!\!\!/$ is W-twisted Dirac operator on $E := S \otimes W$.

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$$\int_{(X/AN)^x} c_x \frac{\hat{A}((X/AN)^x)\,\mathrm{ch}([W_{AN}|_{\mathrm{supp}(\chi_x)}](x))e^{c_1(L|_{(X/AN)^x})}}{\det(1 - xe^{-R^N/2\pi i})^{1/2}}.$$

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$$\int_{(X/AN)^x} c_x \frac{\hat{A}((X/AN)^x)\,\mathrm{ch}([W_{AN}|_{\mathrm{supp}(\chi_x)}](x))e^{c_1(L|_{(X/AN)^x})}}{\det(1 - xe^{-R^N/2\pi i})^{1/2}}.$$

• *If P is not a maximal cuspidal parabolic subgroup or x does not lie in a compact subgroup of G, then $\Phi_x^P(\mathrm{Ind}(\not{D}))$ vanishes.*

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Using $\Delta_T^M(t)\Phi_{P,x}$, we can define a morphism

$$\mathcal{F}^T : K_0(C_r^*(G)) \to \mathfrak{Rep}(K).$$

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Structure of $C_r^*(G)$

Theorem (Wassermann, Clare-Crisp-Higson)

The $C_r^*(G)$ and also $\mathcal{C}(G)$ have the following decomposition,

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The component $C_r^*(G)_{[P,\sigma]}$ is Morita equivalent to

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For example, for $SL_2(\mathbb{R})$,

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Thank you for your attention!