Twisted real structures for spectral triples

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Section 1

Origins and motivation

Spectral triples

A spectral triple is $(A, \mathcal{H}, D)$ with A represented on $\mathcal{H}$ as bounded operators and D an unbounded self-adjoint operator on $\mathcal{H}$ with compact resolvent and $[D, a] \subseteq B(\mathcal{H})$.
If $\mathcal{H}$ admits a $\mathbb{Z}_2$-grading, it can be made *even* by incorporating a grading operator γ with $\gamma D = -D\gamma$.

Idea: enrich Gelfand-Naimark duality $(A, \mathcal{H}) \leftrightsquigarrow$ *topology* to give $(A, \mathcal{H}, D) \leftrightsquigarrow$ *geometry*.

Canonical spectral triple

Any closed spin^c manifold M can be described by the spectral triple

$$(C^\infty(M), L^2(M, S), i\nabla\!\!\!/).$$

Origins and motivation

Real spectral triple

A real spectral triple is $(A, \mathcal{H}, D, J, (\gamma))$ with $(A, \mathcal{H}, D, (\gamma))$ a(n even) spectral triple and

1. J an antilinear isometry of $\mathcal{H}$ (the real structure);
2. $\left[a, JbJ^{-1}\right] = 0$ for all $a, b \in A$;
3. $\left[[D, a], JbJ^{-1}\right] = 0$ for all $a, b \in A$ [1C];
4. $$J^2 = \varepsilon 1, \quad JD = \varepsilon' DJ \quad J\gamma = \varepsilon'' \gamma J$$

   where

j	0	1	2	3	4	5	6	7
ε	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
ε'	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$
ε''	$+$		$-$		$+$		$-$	

Origins and motivation
Examples of spectral triples

Noncommutative (2-)torus

$(C(T^2_\theta), L^2(T^2_\theta) \otimes \mathbb{C}^2, D, J, \gamma)$ with

- $C(T^2_\theta)$ the universal C*-algebra generated by the unitaries U, V with $VU = e^{-2\pi i\theta} UV$;
- $L^2(T^2_\theta)$ the norm-completion of $C(T^2_\theta)$ coming from the trace;
- $D = -i(\sigma_1 \delta_1 + \sigma_2 \delta_2)$;
- $J = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}$ for J_0 the Tomita-Takesaki operator;
- γ the natural $\mathbb{Z}_2$-grading.

There are many more examples of spectral triples, including (but not limited to)

- Fuzzy spaces
- q-deformations
- Moyal spaces
- Crossed product extensions
- Spaces of leaves of a foliation
- Space of Penrose tilings
- Duals of discrete groups
- ...

However, often these are not real.

Origins and motivation

A conformal transformation for spectral triples

Conformal transformation

Consider a spin manifold (M, g). A conformal transformation of the metric maps $g \mapsto \lambda^2 g$ for $\lambda \in C^\infty(\mathbb{R})$. The corresponding transformation of the Dirac operator is $\partial\!\!\!/ \mapsto \lambda^{-\frac{1}{2}} \partial\!\!\!/ \lambda^{-\frac{1}{2}}$.

In the case of A noncommutative, $D \mapsto kDk$ for $k \in A^+$ does *not* give a spectral triple for A, but rather a *twisted* spectral triple.

Conformally transformed real spectral triple

Let $(A, \mathcal{H}, D, J)$ be a real spectral triple and define the operator $v := k^{-1} J k J^{-1}$ for $k \in A^{+}$ and $D_k := J k J^{-1} D J k J^{-1}$.
Then $(A, \mathcal{H}, D_k)$ is a spectral triple, but J satisfies modified relations

- $J b J^{-1} [D_k, a] = [D_k, a] J v^2 b v^{-2} J^{-1}$;
- $v J D_k = \pm D_k J v$.

Note that this gives the correct transformation when $A = C^{\infty}(M)$.

Origins and motivation

Spectral triple with twisted real structure

Definition (Spectral triple with twisted real structure)

Let $(A, \mathcal{H}, D)$ with rep π be a spectral triple.
Let $\nu \colon \mathcal{H} \to \mathcal{H}$ give an automorphism of A via $\bar{\nu}(A) := \nu\pi(A)\nu^{-1} \simeq \pi(A)$, and let $J \colon \mathcal{H} \to \mathcal{H}$ be an antilinear isometry.
Then (J, ν) is a ν–twisted real structure for $(A, \mathcal{H}, D)$ if

- $J^2 = \varepsilon' 1, \quad \nu J D = \varepsilon' D J \nu;$
- $\left[a, JbJ^{-1}\right] = 0$ for all $a, b \in A$;
- $\left[[D, a], JbJ^{-1}\right]_{\bar{\nu}^{-2}} = 0$ for all $a, b \in A$;
- $\nu J \nu = J,$

for $\varepsilon, \varepsilon' \in \{+, -\}$.
If $(A, \mathcal{H}, D)$ admits a grading γ, we also require $\gamma \nu J = \varepsilon'' \nu J \gamma$ and $\gamma \nu^2 = \nu^2 \gamma$ for $\varepsilon'' \in \{+, -\}$.

Section 2

Hodge–de Rham spectral triples

Spin vs. Hodge

For a real spectral triple $(A, \mathcal{H}, D, J)$, let $\mathcal{C}\ell_D(A) \subseteq B(\mathcal{H})$ be the C*-subalgebra generated by A and $[D, A]$.

Definition

We call a spectral triple

$$\text{spin} \iff \mathcal{C}\ell_D(A) \overset{M}{\sim} \bar{A}$$

$$\text{Hodge} \iff \mathcal{C}\ell_D(A) \overset{M}{\sim} \mathcal{C}\ell_D(A)$$

0C: $JA^* J^{-1} \subseteq A'$

1C: $J\Omega^1_D(A)^* J^{-1} \subseteq A'$

2C: $J\Omega^1_D(A)^* J^{-1} \subseteq \Omega^1_D(A)'$

0C and 1C $\longleftrightarrow$ $J\mathcal{C}\ell_D(A)^* J^{-1} \subseteq A'$

0C, 1C and 2C $\longleftrightarrow$ $J\mathcal{C}\ell_D(A)^* J^{-1} \subseteq \mathcal{C}\ell_D(A)'$

Hodge-de Rham spectral triples
Hodge-de Rham spectral triple

Hodge-de Rham spectral triple

A (compact orientable) Riemannian manifold admits the spectral triple

$$(C^\infty(M), L^2(\textstyle\bigwedge_{\mathbb{C}}^\bullet T^*M), -i(d - d^*)).$$

Can equip with a grading: for α_k a k-form,

$$\gamma\alpha_k = (-1)^k\alpha_k.$$

Hodge–de Rham spectral triples
Real structure

Two natural antilinear isometries on $L^2(\bigwedge^\bullet_{\mathbb{C}} T^*M)$:

$$C\alpha = \bar{\alpha},$$
$$J\alpha = \alpha^*.$$

C is a real structure but does not implement the Hodge condition.
J is not a real structure but does implement the Hodge condition.

$$J(-i(d-d^*))J^{-1} = i(d+d^*)\gamma \neq \mp i(d-d^*).$$

However…

Hodge-de Rham spectral triples

Twisted real structure

Proposition (D'Andrea, Dąbrowski, M. '21)

J, along with the operator

$$v\alpha_k = i^{k(k+1)}\alpha_k,$$

are together a twisted real structure.

In particular,

$$vJ(-i(d-d^*)) = -i(d-d^*)Jv;$$

Twisted-1C satisfied because $v^2 f v^{-2} = f$ (twist is *mild*) and J satisfies 1C;
$vJv = J$ because $Ji^{k(k+1)} = i^{-k(k+1)}J$;
v, γ, J all commute.

Section 3

Gauge transformations
First-order condition

Result (Chamseddine, Connes, van Suijlekom '13)

It is possible to define gauge theory in NCG without 1C. In this case, fluctuations look like

$$D \mapsto D + a[D, b] + a^\circ[D, b^\circ] + aa^\circ[[D, b], b^\circ].$$

The almost-commutative spectral triple coming from $A_F = \mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$ without 1C gives $SU(2) \times SU(2) \times SU(4)$ Pati-Salam-like models.

Question: Is it possible to accomplish something similar with only a *weakened* 1C?

Gauge transformations

Gauge theory in NCG

How to think about gauge theory in NCG: $(\mathcal{A}, \mathcal{H}, D, J)$ and Morita (self-)equivalences.

1. Let $\mathcal{E}$ be a full fin. gen. proj. (right) Hilbert $\mathcal{A}$-module. Then $\mathcal{A} \overset{M}{\sim} \mathrm{End}_{\mathcal{A}}(\mathcal{E})$.
2. Let $\mathcal{A}$ be rep'd on $\mathcal{H}$. Then $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ acts on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$.
3. To extend D from $\mathcal{H}$ to $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$, one needs to introduce a connection ∇ s.t. $D'(e \otimes \psi) = e \otimes D\psi + \nabla(e)\psi$

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$$

$$\nabla(ea) = \nabla(e)a + e \otimes \delta(a)$$

Gauge transformations

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4. $\mathcal{A} \overset{M}{\sim} \mathcal{B} \iff \mathcal{B} \overset{M}{\sim} \mathcal{A}$ so we can also consider a *left* Hilbert $\mathcal{A}$-module $\bar{\mathcal{E}}$ s.t. $\mathrm{End}_{\mathcal{A}}(\bar{\mathcal{E}})$ acts on $\mathcal{H} \otimes_{\mathcal{A}} \bar{\mathcal{E}}$
5. Extend D to $\mathcal{H} \otimes_{\mathcal{A}} \bar{\mathcal{E}}$ by $D''(\psi \otimes \bar{e}) = D\psi \otimes \bar{e} + \psi\nabla^\circ(\bar{e})$ where $\nabla^\circ \colon \bar{\mathcal{E}} \to \Omega^1_D(\mathcal{A}^\circ) \otimes_{\mathcal{A}} \bar{\mathcal{E}}$
6. Recognise left- and right-modules are compatible, *i.e.* we have $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \bar{\mathcal{E}}$ and $D'''(e \otimes \psi \otimes \bar{f}) = e \otimes D\psi \otimes \bar{f} + \nabla(e)\psi \otimes \bar{f} + e \otimes \psi\nabla^\circ(\bar{f})$
7. Observe that $\mathcal{A} \overset{M}{\sim} \mathcal{A}$. In this case, $\mathcal{E} = \bar{\mathcal{E}} = \mathcal{A}$ and

$$\nabla(a)\psi = [D, a]\psi + \alpha a\psi,$$
$$\psi\nabla^\circ(b) = \left[D, JbJ^{-1}\right]\psi \pm J\alpha J^{-1} JbJ^{-1}\psi,$$

$\alpha \in \Omega^1_D(\mathcal{A})$, so $D''' = D + \alpha \pm J\alpha J^{-1} := D_\alpha$ when $\psi b = JbJ^{-1}\psi$.

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8. Define $\mathcal{U}(\mathcal{E}) := \{u \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}) : uu^* = u^*u = 1\}$. A gauge transformation is the action of u on $\mathcal{E}$ (and on $\bar{\mathcal{E}}$)
9. When $\mathcal{E} = \bar{\mathcal{E}} = \mathcal{A}$, identify the gauge transformations as:

$$\begin{aligned}
\psi &\mapsto u\psi u^*, \ \psi \in \mathcal{H}; \\
a &\mapsto uau^*, \ a \in \mathcal{A}; \\
\alpha &\mapsto u\alpha u^* + u[D, u^*] =: \alpha^u; \\
D_\alpha &\mapsto D_{\alpha^u}
\end{aligned}$$

for $u \in \mathcal{U}(\mathcal{A})$.

Warning! We would like to have a twisted real structure, but for $D^u = D + u[D, u^*] \pm Ju[D, u^*]J^{-1}$,

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Gauge transformations

Gauge transformations with twisted real structures

What we want is

$$D_\alpha = D + \alpha \pm \nu J \alpha J^{-1} \nu,$$

since then $\nu J D_\alpha = \pm D_\alpha J \nu$. But can we get it via Morita self-equivalences?

First, observe that if $\alpha = \sum a[D, b]$, then

$$\pm \nu J \alpha J^{-1} \nu = \sum J \nu^{-1} a \nu J^{-1} \left[D, J \nu b \nu^{-1} J^{-1}\right]_{\bar{\nu}^2}$$

This is a twisted 1-form! So there is no need to change $\Omega^1_D(\mathcal{A})$, but $\Omega^1_D(\mathcal{A}^\circ) \rightsquigarrow \Omega^1_D(\mathcal{A}^\circ)_{\bar{\nu}^2}$ is twisted (and uses a different rep).

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$$\nabla(a)\psi = [D, a]\psi + \alpha a \psi,$$
$$\psi \tilde{\nabla}^\circ(b) = \left[D, J\nu b \nu^{-1} J^{-1}\right]_{\bar{\nu}^2} \psi + \nu J \alpha J^{-1} \nu J \nu b \nu^{-1} J^{-1} \psi,$$

$\alpha \in \Omega^1_D(\mathcal{A})$, so $D''' = D + \alpha \pm \nu J \alpha J^{-1} \nu := D_\alpha$ when $\psi b = J\nu b \nu^{-1} J^{-1} \psi$.

Lemma (Landi, Martinetti '18; M., Dąbrowski)

$D'' = (1 \otimes \tilde{v}^2) \circ (D_{\mathcal{H}} + \tilde{\nabla}^\circ)$ is a well defined Dirac operator on $\mathcal{H} \otimes_{\mathcal{A}} \bar{\mathcal{E}}$.

Choose the right $\mathcal{A}$-action $\psi a := J v a^* v^{-1} J \psi$ and let $\tilde{v}(a\bar{e}) = v a v^{-1} \bar{e}$.

Define $\tilde{\delta}^\circ(a) = D J v a^* v^{-1} J^{-1} - J v^{-1} a^* v J^{-1} D$. Then $\tilde{\delta}^\circ$ is a derivation of A for $a \cdot \tilde{\alpha}^\circ \cdot b := J v^{-1} b^* v J \tilde{\alpha}^\circ J v a^* v^{-1} J$.

Define a connection $\tilde{\nabla}^\circ \colon \bar{\mathcal{E}} \to \Omega^1_D(\mathcal{A}^\circ)_{\tilde{v}^2} \otimes_{\mathcal{A}} \bar{\mathcal{E}}$ such that $\tilde{\nabla}^\circ(a\bar{e}) - a \cdot \tilde{\nabla}^\circ(\bar{e}) = \tilde{\delta}^\circ(a) \otimes \bar{e}$.

Extend $\tilde{\nabla}^\circ$ to $\mathcal{H} \otimes_{\mathbb{C}} \bar{\mathcal{E}}$ using $\psi \tilde{\nabla}^\circ(a\bar{e}) - \psi a \tilde{\nabla}^\circ(\bar{e}) = \tilde{\delta}^\circ(a) \psi \otimes \bar{e}$.

Then $D''(\psi \otimes a\bar{e}) - D''(\psi a \otimes \bar{e}) = 0$. When $\bar{\mathcal{E}} = \mathcal{A}$, D'' on $\mathcal{H}$ is given by $D + \tilde{\alpha}^\circ = D \pm v J \alpha J^{-1} v$.

When $\bar{\mathcal{E}} = \mathcal{A}$, $\tilde{\nabla}^\circ = \tilde{\delta}^\circ + \tilde{\alpha}^\circ$ with $\tilde{\alpha}^\circ(a) = a \cdot \tilde{\alpha}^\circ$ for arbitrary $\tilde{\alpha}^\circ \in \Omega^1_D(\mathcal{A}^\circ)_{\tilde{v}^2}$. Then

$$
\begin{aligned}
D''(\psi \otimes a) &= (1 \otimes \tilde{v}^2) \circ (D_{\mathcal{H}} + \tilde{\nabla}^\circ)(\psi \otimes a) \\
&= (1 \otimes \tilde{v}^2)(D\psi \otimes a) + (1 \otimes \tilde{v}^2)(\tilde{\delta}^\circ + \tilde{\alpha}^\circ)(\psi \otimes a) \\
&= (D\psi)\tilde{v}^2(a) \otimes 1 + (1 \otimes \tilde{v}^2)(\tilde{\delta}^\circ(a))\psi \otimes 1) \\
&\qquad + (1 \otimes \tilde{v}^2)(\tilde{\alpha}^\circ J v a^* v^{-1} J^{-1} \psi \otimes 1) \\
&= J v^{-1} a^* v J^{-1}(D\psi) \otimes 1 + \big(D J v a^* v^{-1} J^{-1} \psi \otimes 1 \\
&\qquad - J v^{-1} a^* v J^{-1} D\psi \otimes 1\big) + \tilde{\alpha}^\circ J v a^* v^{-1} J^{-1} \psi \otimes 1 \\
&= D J v a^* v^{-1} J^{-1} \psi \otimes 1 + \tilde{\alpha}^\circ J v a^* v^{-1} J^{-1} \psi \otimes 1
\end{aligned}
$$

Gauge transformations
Operator implementation

Note that $\tilde{\alpha}^\circ$ is compatible with the definition of α^u since $\pm vJ\alpha^u J^{-1}v = (\tilde{\alpha^u})^\circ = u \cdot \tilde{\alpha}^\circ \cdot u^* + \tilde{\delta}^\circ(u) \cdot u^* = (\tilde{\alpha}^\circ)^u$.

Because of the new right action, we now have

$$u\psi u^* = uJvuv^{-1}J^{-1}\psi =: V\psi.$$

For convenience, we also define

$$W := uJv^{-1}uvJ^{-1}.$$

These are not unitary for general v!

Using $V = uJvuv^{-1}J^{-1}$, $W = uJv^{-1}uvJ^{-1}$ we find

$$uau^* = VaV^{-1} = WaW^{-1}$$
$$D^u_\alpha = WD_\alpha V^{-1}$$

$$\begin{cases} vJ = WvJW^{-1} \\ Jv = VJvV^{-1} \end{cases}$$

Some observations...

- V and W are unitary when v is unitary.
- $v = WvV^{-1}$ and $J = VJW^{-1}$ when v is unitary and self-adjoint.
- If v is unitary and self-adjoint, then $V = W$.

D^u is not automatically self-adjoint; however, taking $v = \pm v^*$ is sufficient for $D^u = (D^u)^*$.

Gauge transformations

Dynamics and the physical action functionals

Assuming $\nu = \pm\nu^*$...

The bilinear form $\mathfrak{A}_D(\psi, \varphi) := \langle J\psi, D\varphi\rangle$ is not (anti)symmetric, so it is necessary to modify it. The correct modification is given by

$$\tilde{\mathfrak{A}}_D(\psi, \varphi) := \langle J\nu\psi, D\varphi\rangle$$

which is not only (anti)symmetric but also gauge-invariant:

$$\tilde{\mathfrak{A}}_{D^u}(\psi^u, \varphi^u) = \tilde{\mathfrak{A}}_D(\psi, \varphi).$$

Thus, it can be used to define the fermionic action functional $S_F[D_\alpha, \psi] = \tilde{\mathfrak{A}}_{D_\alpha}(\tilde{\psi}, \tilde{\psi})$.

Suppose $D\psi = \lambda\psi$. Then

$$D^u \psi^u = WDV^{-1} V\psi = \lambda W\psi \neq \lambda V\psi.$$

This is a problem!

The bosonic action is given by

$$S_B[D_\alpha] = \mathrm{Tr}\left(f(D_\alpha/\Lambda)\right).$$

If D has a different spectrum than D^u, then S_B cannot be gauge-invariant!

However, this problem disappears when $W = V$, *i.e.*, when $\nu = \pm\nu^*$ and $\nu^* = \pm\nu^{-1}$.

Section 4

Multitwists and other work

Multitwisted real structure

Definition (Spectral triple with multitwisted real structure)

Let $(A, \mathcal{H}, D)$ be a spectral triple, and $D = \sum_{i \in I} D_i$.
Let $\nu_i : \mathcal{H} \to \mathcal{H}$ give automorphisms of $B(\mathcal{H})$ via $\bar{\nu}_i(B(\mathcal{H})) := \nu_i B(\mathcal{H}) \nu_i^{-1}$, and let $J : \mathcal{H} \to \mathcal{H}$ be an antilinear isometry.
Then $(J, \{\nu_i\})$, $i \in I$ is a multitwisted real structure for $(A, \mathcal{H}, D)$ if

- $J^2 = \varepsilon' 1, \quad \nu_i J D_i = \varepsilon' D_i J \nu_i$ for each $i \in I$;
- $\left[a, J\bar{\nu}_i(b)J^{-1}\right] = 0$ and $\left[a, J\bar{\nu}_i^{-1}(b)J^{-1}\right] = 0$ for all $a, b \in A$ and for each $i \in I$;
- $\left[[D_i, a], J\bar{\nu}_i(b)J^{-1}\right]_{\bar{\nu}_i^{-2}} = 0$ for all $a, b \in A$ and for each $i \in I$;
- $\nu_i J \nu_i = J$ for each $i \in I$,

for $\varepsilon, \varepsilon' \in \{+, -\}$.
If $(A, \mathcal{H}, D)$ admits a grading γ, we also require $\gamma \nu_i J = \varepsilon'' \nu_i J \gamma$ and $\gamma \nu_i^2 = \nu_i^2 \gamma$ for each $i \in I$, for $\varepsilon'' \in \{+, -\}$.

Multitwists and other work

Asymmetric noncommutative torus

Take the noncommutative torus and transform the Dirac operator

$$\sigma^1\delta_1 + \sigma^2\delta_2 \mapsto \sigma^1\delta_1 + JkJ^{-1}\sigma^2\delta_2 JkJ^{-1}.$$

One obtains an asymmetric noncommutative torus from a symmetric one via (multi)conformal transformation.

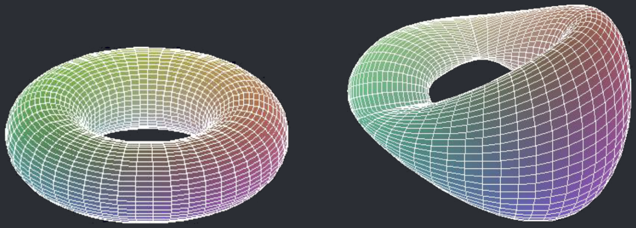

Figure: L. Dąbrowski & A. Sitarz, *Asymmetric noncommutative torus*, SIGMA, (2015).

Multi-twists and other work

Spectral Pati-Salam

Returning to our earlier question...
Is it possible to define Pati-Salam-like models satisfying twisted-1C?

Issue: When $\nu = \pm\nu^{-1}$, twisted-1C becomes 1C, so there's no weakening and A_F breaks to $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.
Idea: Use multi-twists – even when $\nu_i^2 = 1$, $J\bar{\nu}_i(A)J^{-1} \not\simeq A$, so twisted-1C $\neq$ 1C.
Result: A_F still breaks to $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ – twist isn't strong enough!

Multi-twists and other work

Ongoing and future work

Where to now? A few ideas...

- $\nu = \nu^{-1} = \nu^*$ in $\tilde{\mathfrak{A}}_D(\psi, \varphi)$ resembles a fundamental symmetry. Idea: use a (multi)twisted real structure to define Wick rotation from Euclidean to Lorentzian signature for spectral triples
- Something like a twisted real structure occurs naturally for modular-type twisted spectral triples; a clue for twisted real structures for twisted spectral triples? Connection to Tomita-Takesaki operator?
- Twisted real structures for more classes of spectral triples (quantum groups, ...)

Section 5

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