Higher Kazhdan projections, ℓ^2-Betti numbers & the coarse Baum-Connes conjecture

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Kazhdan's property (T)

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Theorem (Delorme-Guichardet)

Let G be a discrete group. TFAE.

- G has property (T)
- $H^1(G, \mathcal{H})$ is reduced for every unitary representation $(\pi, \mathcal{H})$.
- $H^1(G, \mathcal{H}) = 0$ for every unitary representation $(\pi, \mathcal{H})$.

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A group G of type F_{n+1} has property (T_n) if $H^{n+1}(G, \mathcal{H})$ is reduced for every unitary representation $(\pi, \mathcal{H})$,

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Remark
In higher degrees : vanishing $\not\Leftrightarrow$ reducedness

For Γ a lattice in $\mathrm{PGL}_{n+1}(\mathbb{Q}_p)$, where $n \geq 2$ and p is a sufficiently large prime, Dymara-Januszkiewicz showed that $H^n(\Gamma, \mathcal{H})$ is reduced for all $(\pi, \mathcal{H})$ but not all of them vanish.

Operator algebraic generalisation

Besides the aforementioned cohomological characterisations of property (T), there is a C^*-algebraic characterisation of property (T).

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Characterisation

A group G has property (T) iff there exists a projection $p \in \mathrm{C}^*_{\max} G$ whose image under any unitary representation $(\pi, \mathcal{H})$ of G is the orthogonal projection $\mathcal{H} \to \mathcal{H}^{\pi(G)}$ onto the fixed vectors.

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- This projection is called *Kazhdan projection*.
- The Kazhdan projection is unique and non-zero inside $\mathrm{C}^*_{\mathrm{max}} G$.
- For an infinite group G, Kazhdan projection in $\mathrm{C}^*_{\mathrm{red}} G$ is always zero.
- Existence of this projection violates a certain method of proof for the Baum-Connes conjecture.
- Aiming to provide new counterexamples, we introduce a higher degree analogue of this projection.

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$$\ker \pi(\Delta) = \mathcal{H}^{\pi(G)}.$$

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Therefore we have

$$\ker \pi(\Delta) = \mathcal{H}^{\pi(G)} = H^0(G, \mathcal{H})$$

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Take the free resolution associated to a BG for the trivial $\mathbb{Z}G$-module $\mathbb{Z}$

$$\cdots \to \mathbb{Z}G^{\oplus k_2} \to \mathbb{Z}G^{\oplus k_1} \to \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

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Hodge-de Rham isomorphism

$$\tilde{H}^n(G, \mathcal{H}) \cong \ker \pi(\Delta_n).$$

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Let G be a group. Let $(\pi, \mathcal{H})$ be a unitary representation of G. The higher Kazhdan projection in degree n associated with π is the orthogonal projection $p_n \colon \mathcal{H}^{\oplus k_n} \to \ker \pi(\Delta_n)$.

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Remark

- This projection always lives in the matrices in the von Neumann algebra generated by $\pi(G)$.
- Assuming spectral gap, the entries lie in the C^*-algebra generated by $\pi(G)$.
- It is the spectral projection $p_n = \lim_{t\to\infty} e^{-t\pi(\Delta_n)}$.

ℓ^2-Betti numbers

- $LG = \overline{\mathbb{C}G}^{SOT} \subseteq \mathcal{B}(\ell^2 G)$ group von Neumann algebra of G
- $\tau \colon LG \to \mathbb{C}$ canonical trace defined by $\tau(\sum_{\text{finite}} c_g g) = c_e$

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- $\tau \colon LG \to \mathbb{C}$ canonical trace defined by $\tau(\sum_{\text{finite}} c_g g) = c_e$
- n-th ℓ^2-Betti number $\beta^n_{(2)}(G) = \dim_{LG} \tilde{H}^n(G, \ell^2 G) \in [0, \infty]$

$$\tilde{H}^n(G, \ell^2 G) \cong p_n(\ell^2 G^{\oplus k_n}) \qquad \text{right LG-module}$$

$$\beta^n_{(2)}(G) = \dim_{LG} \tilde{H}^n(G, \ell^2 G) = \dim_{LG} p_n(\ell^2 G^{\oplus k_n}) = (Tr \otimes \tau)(p_n)$$

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Proposition

Assume $\lambda(\Delta_n)$ *has spectral gap so that* p_n *belongs to* $\mathrm{M}_{k_n}(\mathrm{C}^*_{red}G)$.

Then we have that

$$\tau_*([p_n]) = \beta^n_{(2)}(G).$$

In particular if $\beta^n_{(2)}(G) \neq 0$, *then* $[p_n] \neq 0$ *in* $K_0(\mathrm{C}^*_{red}G)$.

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Conjecture (Modified Trace Conjecture, Lück, 2002)
Let G *be a discrete group. We have*

$$\tau_*(K_0(\mathrm{C}^*_{red}G)) \subseteq \langle \frac{1}{|F|} \mid F \leq^{finite} G\rangle_{ring} \subseteq \mathbb{Q}.$$

If the Baum-Connes assembly map for G is surjective, then the conjecture is a theorem.

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Assume $\lambda(\Delta_n)$ has spectral gap so that p_n lives in $\mathrm{M}_{k_n}(\mathrm{C}^*_{red}G)$. Surjectivity of the Baum-Connes assembly map implies that $\beta^n_{(2)}(G) \in \mathbb{Q}$. In particular if G is torsion-free, it is an integer.

Atiyah conjecture

Atiyah introduced L^2-Betti numbers for compact manifolds where later generalised to finite CW-complexes.
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Conjecture (Bounded, Strong Atiyah conjecture, Lück-Schick)
Let G be a group with a bound on the orders of finite subgroups. For every $n \in \mathbb{N}$ and every $T \in \mathrm{M}_n(\mathbb{C}G)$, we have

$$\dim_{LG}(\ker(T)) := \mathrm{Tr} \otimes \tau(\mathrm{p}_{\ker T}) \in \langle \frac{1}{|F|} \mid F \leq^{finite} G \rangle_{group} \subseteq \mathbb{Q}$$

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- $G \leq \mathrm{PGL}_3(\mathbb{Q}_p)$, p large prime. $\rightsquigarrow p_2 \in \mathrm{M}_{k_2}(\mathrm{C}^*_{\max}(G))$ is non-zero (Dymara-Januszkiewicz)

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Coarse Baum-Connes conjecture, Roe, 1993

For all X with bounded geometry the assembly map is an isomorphism

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The coarse Baum-Connes conjecture does *have* counterexamples, but also *lots of* confirmed cases.

Application to the coarse Baum-Connes conjecture

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Theorem (Li-Nowak-P.)

Let G be an exact residually finite group of type F_{n+1}. Let $N = \{N_i\}_i$ be a filtration of finite index normal subgroups of G. Let $\pi = \oplus_i \lambda_i$. Assume that $\pi(\Delta_n)$ has a spectral gap such that p_n belongs to $\mathrm{M}_{k_n}(\mathrm{C}^*_N G)$. If the coarse Baum-Connes assembly map for the box space $Y = \coprod G/N_i$ of G is surjective, then

$$\beta^n_{(2)}(N_i) = \beta^n(N_i)$$

for all but finitely many i.

Application to the coarse Baum-Connes conjecture

- $\beta^n(G) = \dim_{\mathbb{C}} H^n(G, \mathbb{C}) \in \mathbb{N}$.
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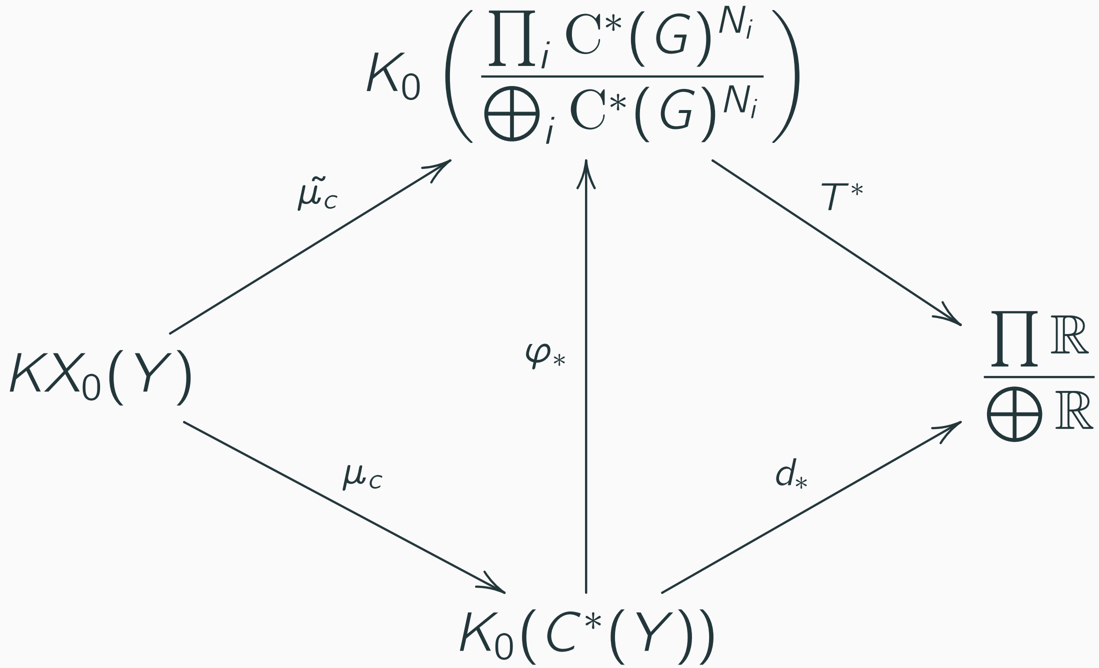
$\rightsquigarrow$ strategy to find counterexamples to the conjecture

On the proof

The proof relies on work of Higson and Willett-Yu.



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When compared, our result has a stronger implication:

$$\beta^n_{(2)}(G) = \frac{\beta^n(N_i)}{[G : N_i]}$$

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- A sequence $(X_i)_{i>0}$ of finite, connected, d-regular graphs is a family of expanders, if $\lim_{i\to\infty} |V_{X_i}| = \infty$ and there is uniform spectral gap.

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- High-dimensional expanders are simplicial complexes that can be viewed as higher-dimensional analogs of expander graphs.
- There are several approaches to defining high-dimensional expansion, and one of them is via uniform spectral gaps for the Laplacian in simplicial cohomology.
- In that sense, an n-dimensional spectral expander is a sequence of finite simplicial complexes $(S_i)_{i>0}$ such that a family of operators of type Δ_i^+ has uniform spectral gap.

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Question
Under which conditions do high-dimensional spectral expanders provide counterexamples to the coarse Baum-Connes conjecture?

Thanks for listening!