Modular Spectral Triples arising from type III representations of the noncommutative torus

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abstract

We discuss the explicit construction of non type II_1 representations and relative modular spectral triples of the noncommutative 2-torus $\mathbb{A}_\alpha$, provided α is a (special kind of) Liouville number, where the nontrivial modular structure plays a crucial role.

For such representations, we briefly discuss the appropriate Fourier analysis, by proving the analogous of many of the classical known theorems in Harmonic Analysis such as the Riemann-Lebesgue Lemma, the Hausdorff-Young Theorem, and the L^p-convergence results such as these associated to the Cesaro means (*i.e.* the Fejer theorem) and the Abel means reproducing the Poisson kernel. We show how those Fourier transforms "diagonalise" appropriately some examples of the Dirac operators associated to the previous mentioned spectral triples.

Finally, we outline the description (which is contained in a work in progress) of a deformed generalisation of "Fredholm module", *i.e.* a suitably deformed commutator of the "phase" of the involved Dirac operator with elements of a subset which, depending on the cases under consideration, is either a dense $*$-algebra or an essential operator system, of the (representation of the) C^*-algebra associated to the quantum manifold under consideration. This definition of deformed Fredholm module is new even in the usual cases associated to a trace and could provide other, hopefully interesting, applications.

The present talk is based on the following papers:

[1] F. Fidaleo and L. Suriano: *Type III representations and modular spectral triples for*

the noncommutative torus, J. Funct. Anal. **275** (2018), 1484-1531.

[2] F. Fidaleo: *Fourier analysis for type III representations of the noncommutative torus*, J. Fourier Anal. Appl. **25** (201), 2801-2835.

[3] F. Ciolli and F. Fidaleo: *Type III modular spectral triples and deformed Fredholm modules*, work in preparation.

introduction

For any irrational rotation number α, it is well known that the noncommutative torus $\mathbb{A}_\alpha$ must have representations π such that the generated von Neumann algebra $\pi(\mathbb{A}_\alpha)''$ is of type III.

For $\mathbb{A}_{2\alpha}$ (the factor 2 is inserted only for matter of convenience), we briefly outline how to construct such no II_1 representations, at least when α is Liouville number.

The idea is to look at C^∞ diffeomorphisms f of the unit circle $\mathbb{T}$ with rotation number, $\rho(f) = 2\alpha$. By Denjoy theorem, f is conjugate to the rotation $R_{2\alpha}$ of the angle $4\pi\alpha$:

$$f = h_f \circ R_{2\alpha} \circ h_f^{-1},$$

where h_f is a (essentially unique) homeomorphism of $\mathbb{T}$. If α is diophantine, then h_f is necessarily smooth and therefore our construction provides still the type II_1 hyperfinite factor. If α is Liouville, things go differently. In this situation, we are looking at diffeomorphisms as above for which the unique probability measure, invariant under the rotations, satisfies

$$m \circ h_f^{-1} \perp m,$$

$m = \frac{\mathrm{d}\theta}{2\pi}$ being the Haar measure on $\mathbb{T}$. Such diffeomorfisms are explicitly constructed in "S. Matsumoto: Nonlinearity 26 (2013), 1401-1414" for any prescribed Krieger-Araki-Woods ratio-set.[*]

For our construction, the probability measure $\mu := m \circ \hbar_f$ plays a crucial role. It is then possible to exhibit a state ω_μ canonically associated to the measure μ, and our non type II_1 representations π_{ω_μ} produce hyperfinite von Neumann factors isomorphic to the crossed product

$$\pi_{\omega_\mu}(\mathbb{A}_{2\alpha})'' \sim L^\infty(\mathbb{T}, \mu) \ltimes_{R_{2\alpha}} \mathbb{Z} \sim L^\infty(\mathbb{T}, m) \ltimes_\rho \mathbb{Z}.$$

Here, β is the dual action of f on functions: $\beta(g) := g \circ f$.

[*] For the dynamical system (X, T, ν) for which the probability measure ν (supposed to be atom-less) is quasi-invariant and ergodic for the action of the automorphism T, the ratio-set $r(T)$ determines the type of the factor $L^\infty(X, \nu) \ltimes_T \mathbb{Z}$, provided it is not of type II_1.

For such representations, we also construct a one-parameter family of modular spectral triple, one for each $\eta \in [0,1]$, whose twists are constructed by taking into account of the modular data, provided α is a Liouville number satisfying a faster approximation property by rationals.[†]

By considering the von Neumann algebra $M := \pi_{\omega_\mu}(\mathbb{A}_{2\alpha})''$ equipped with the "measure" $\omega := \langle \cdot \xi_{\omega_\mu}, \xi_{\omega_\mu} \rangle$, we consider the various noncommutative L^p spaces $L^p(M)$, and define the Fourier transform(s). At level of Hilbert spaces (*i.e.* $L^2(M)$), we show that they "diagonalise" the involved deformed Dirac operators $D^{(\eta)}$, for $\eta = 0, 1/2, 1$, corresponding to the left, symmetric and right embeddings

$$L^\infty\big(\pi(\mathbb{A}_{2\alpha})''\big) \equiv L^\infty(M) \equiv M \\ \hookrightarrow M_* \equiv L^1(M) \equiv L^1\big(\pi(\mathbb{A}_{2\alpha})''\big) \,,$$

[†]In our preliminary construction, for such a purpose we consider only the Tomita modular operator Δ, the conjugation J will play a crucial role in a more refined analysis.

of M in M_* associated to the underlying state ω.

In order to exhibit such examples of modular spectral triples associated to type III representations for which the domain of the commutator, deformed by the modular operator, is indeed a dense subalgebra of $\mathbb{A}_{2\alpha}$, we need to suitably change the undeformed Dirac operator

$$D := \begin{pmatrix} 0 & L \\ L^* & 0 \end{pmatrix},$$

where

$$\mathcal{L} = (\partial_1 + \imath\partial_2). \tag{1}$$

Here, the ∂_i are the partial derivatives w.r.t. the angles θ_i, $i = 1, 2$, of functions $f\big(e^{\imath\theta_1}, e^{\imath\theta_2}\big)$ on the 2-torus $\mathbb{T}^2$ which in our context, for $z \in \mathbb{T}$ and $n \in \mathbb{Z}$, assume the form

$$(\partial_1 g)_n(z) := \imath z \frac{\mathrm{d}g_n}{\mathrm{d}z}(z), \quad (\partial_2 g)_n(z) := \imath n g_n(z).$$

The closed operator L arises by representing $\mathcal{L}$ on the Hilbert space $\mathcal{H}_\omega$.

By following the proposal in "A. Connes and W. D van Suijlekom W. D: *Spectral truncations in noncommutative geometry and operator systems*, Commun. Math. Phys., doi:10.1007/s00220-020-03825-x" we can indeed exhibit models of modular spectral triples without changing the intrinsic object (1), but the price to pay is to restrict the domain of the deformed commutator to an "essential" operator space instead of a dense $*$-algebra.[‡]

It is also possible to provide a proposal for a *deformed Fredholm module* which is naturally associated to modular spectral triples, and appears new also for the undeformed case associated to the canonical trace on the noncommutative torus.

[‡]The meaning of "essential" is explained below.

By taking into account of the pivotal model previously described, we end by providing the definition of deformed spectral triples, and the associated Fredholm modules which will describe the general situation.

the noncommutative torus and type III representations

For a fixed $\alpha \in \mathbb{R}$, the *noncommutative torus* $\mathbb{A}_{2\alpha}$ (the factor 2 is pure matter of convenience), *i.e.* that associated with the rotation by the angle $4\pi\alpha$, is the universal C^*-algebra with identity I generated by the commutation relations involving two noncommutative unitary indeterminates U, V:

$$\begin{aligned} &UU^* = U^*U = I = VV^* = V^*V, \\ &UV = e^{4\pi\imath\alpha}VU. \end{aligned} \tag{2}$$

We express $\mathbb{A}_{2\alpha}$ in the so called *Weyl form*. Let $\mathbf{a} := (m, n) \in \mathbb{Z}^2$ be a double sequence of integers, and define

$$W(\mathbf{a}) := e^{-2\pi\imath\alpha mn} U^m V^n, \quad \mathbf{a} \in \mathbb{Z}^2.$$

Obviously, $W(\mathbf{0}) = I$, and the commutation relations (2) become

$$\begin{aligned} &W(\mathbf{a})W(\mathbf{A}) = W(\mathbf{a}+\mathbf{A})e^{2\pi\imath\alpha\sigma(\mathbf{a},\mathbf{A})}, \\ &W(\mathbf{a})^* = W(-\mathbf{a}), \quad \mathbf{a}, \mathbf{A} \in \mathbb{Z}^2, \end{aligned} \tag{3}$$

where the symplectic form σ is defined as

$$\begin{aligned} &\sigma(\mathbf{a}, \mathbf{A}) := (mN - Mn), \\ &\mathbf{a} = (m, n),\ \mathbf{A} = (M, N) \in \mathbb{Z}^2. \end{aligned}$$

We now fix a function $f \in \mathcal{B}(\mathbb{Z}^2)$, which we may assume to have finite support. The element $W(f) \in \mathbb{A}_{2\alpha}$ is then defined as

$$W(f) := \sum_{\mathbf{a}\in\mathbb{Z}^2} f(\mathbf{a})W(\mathbf{a}).$$

The set $\{W(f) \mid f \in \mathcal{B}(\mathbb{Z}^2)\,\text{with finite support}\}$ provides a dense $*$-algebra of $\mathbb{A}_{2\alpha}$. We also recall that $\mathbb{A}_{2\alpha}$ is simple and has a necessarily unique and faithful trace

$$\tau(W(f)) := f(\mathbf{0}), \quad W(f) \in \mathbb{A}_{2\alpha}.$$

Any element $A \in \mathbb{A}_{2\alpha}$ is uniquely determined by the corresponding Fourier coefficients

$$f(\mathbf{a}) := \tau(W(-\mathbf{a})A), \quad \mathbf{a} \in \mathbb{Z}^2.$$

The relations (3) transfer on the generators $W(f)$ as follows

$$W(f)^* = W(f^\star), \quad W(f)W(g) = W(f *_{2\alpha} g),$$

where

$$f^\star(\mathbf{a}) := \overline{f(-\mathbf{a})},$$
$$(f *_{2\alpha} g)(\mathbf{a}) = \sum_{\mathbf{A} \in \mathbb{Z}^2} f(\mathbf{A}) g(\mathbf{a} - \mathbf{A}) e^{-2\pi\imath\alpha\sigma(\mathbf{a},\mathbf{A})}.$$

To construct type III representations, we restrict α to be a so called **L**-numbers (L stands for Liouville), which are numbers, automatically irrational, which admit a "fast approximation by rationals". Here, we report the definition of Liouville and that we call "ultraliouville" numbers.

L A *Liouville number* $\alpha \in (0,1)$ is a real number such that for each $N \in \mathbb{N}$ the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^N}$$

has an infinite number of solutions for $p, q \in \mathbb{N}$ with $(p,q) = 1$.

UL A *Ultra-Liouville number* $\alpha \in (0,1)$ is a real number such that, for each $\lambda > 1$ and $N \in \mathbb{N}$, the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\lambda^{q^N}}$$

again admits infinite number of solutions for p, q as above.

Any **L**-number is irrational, and in addition each **UL**-number is a **L**-number.

Our construction is based upon the explicit knowledge of special C^∞-diffeomorphisms (simply denoted by "diffeomorphisms) of the circle such that $\rho(f) = 2\alpha$. Such diffeomorfisms are constructed in the above mentioned paper of Matsumoto for a prescribed ratio-set F, but without any control on the so-called growth sequence, playing also a crucial role in constructing modular spectral triples.

The *growth sequence* is defined as

$$\Gamma_n(f) := \|Df^n\|_\infty \vee \|Df^{-n}\|_\infty, \quad n \in \mathbb{N}.$$

It was proven (*cf.* "N. Watanabe: Geom. Funct. Anal. 17 (2007), 320-331) that, in our situation, the growth sequence is not too wild: $\Gamma_n(f) = o(n^2)$. We prove that:

(i) for all diffeomorphisms f in Proposition 2.1 of **M** (constructed by fixing any preassigned ratio-set), we have $\Gamma_n(f) = o(n)$;

(ii) if in addition α is a **UL**-number, with the same method in **M** it is possible to construct diffeomorpisms with any preassigned ratio-set, and in addition $\Gamma_n(f) = o(\ln n)$.

Our construction of non type II_1 representations of the noncommutative torus starts by noticing that, if $\nu \in \mathcal{S}(C(\mathbb{T}))$ (*i.e.* ν is a probability measure on the circle) then

$$\omega_\mu(W(f)) := \sum_{m\in\mathbb{Z}} \check{\mu}(m) f(m,0) \qquad (4)$$

defines a state on $\mathbb{A}_{2\alpha}$.[§]

For any diffeomorphism f as before, and $\mu = m \circ h_f$, the crucial fact is that

$$\mu \circ R_\alpha^{2n} \sim \mu, \quad n \in \mathbb{Z}, \qquad (5)$$

[§]The normalised trace corresponds to $\nu = \frac{\mathrm{d}\theta}{2\pi}$.

and therefore the support of ω_μ in the bidual is central: $s(\omega_\mu) \in Z(\mathbb{A}_{2\alpha}^{**})$.[¶]

Put for simplicity $\omega = \omega_\mu$ We notice that the modular structure (i.e. Δ_ω, J_ω) is explicitly expressed by the Radon-Nikodym derivatives $\mathrm{d}\mu \circ R^{2n}/\mathrm{d}\mu$, $n \in \mathbb{Z}$. The representations we are searching for (including the type III ones) are those π_ω associated to the states ω_μ previously described. In this case

$$\pi_\omega(\mathbb{A}_{2\alpha})'' \sim L^\infty(\mathbb{T}, \mathrm{d}\mu) \ltimes_{R_{2\alpha}} \mathbb{Z} \sim L^\infty(\mathbb{T}, \mathrm{d}\theta/2\pi) \ltimes_\beta \mathbb{Z}.$$

is an hyperfinite factor acting in standard form with a copy of L^∞ as MASA, whose type is determined by the Krieger-Araki-Woods *ratio-set*.

[¶]This simply means that the GNS vector ξ_{ω_μ} is also separating.

the Fourier transform

The key-point for the construction of noncommutative L^p spaces is the modular theory. [‖] In fact, for $x \in M$ the map

$$t \in \mathbb{R} \mapsto \sigma_t^\omega(x)\omega \equiv \omega\big(\cdot\,\sigma_t^\omega(x)\big) \in M_*$$

extends to a bounded and continuous map on the strip $\{z \in \mathbb{C} \mid -1 \leqslant \mathrm{Im} z \leqslant 1\}$, analytic in the interior. After putting $L^1_\theta(M) := M_*$, $\theta \in [0,1]$, and $L^\infty_\theta(M) := \iota^\theta_{\infty,1}(M) \sim M$ and checking that the complex interpolation functor C' based on such analytic functions coincides (equal norms) with the standard one C, it is possible to define $L^p_\theta(M) := C_{1/p}(\iota^\theta_{\infty,1}(M), M_*)$, $1 < p < +\infty$. In such a way, for $1 \leqslant p \leqslant q \leqslant +\infty$ there are contractive embeddings

$$\iota^\theta_{q,p} : L^q_\theta(M) \hookrightarrow L^p_\theta(M).$$

[‖]See *e.g.* "H Kosaki: J. Funct. Anal. 56 (1984), 29-78", "M. Terp: J. Operator Theory 9 (1982), 327-360". Another equivalent approach is in "U. Hageerup: Colloques Internationaux C.N.R.S. 274 (1979), 175-184".

We also remark (*cf.* "F. Fidaleo: J. Funct. Anal. 169 (1999), 226-250") that one can equip the L^p-spaces with a canonical operator space structure arising from the canonical embedding $\iota^{\theta}_{\infty,1} : M \to M_*$, which incidentally is also completely bounded. It is possible to show that such an operator space structure coincides with the Pisier OH-structure at level of Hilbert spaces.

Concerning a suitable definition of the Fourier transform for such cases (*i.e.* when the reference measure is not the trace), we have to understand what is the best replacement of the "characters" $e^{\imath(n_1\theta_1+n_2\theta_2)}$, which for the tracial case correspond to $U_1^{n_1}U_2^{n_2}$ (where $U_1 \equiv U$, $U_2 = V$). In the cases we are managing, the choices are reported as follows.

We put $f_k(m,n) := \delta_{m,0}\delta_{n,k}$, $g_l(m,n) := \widehat{(h_f)^l}(m)\delta_{n,0}$, and define $\{u_{kl} \mid k,l \in \mathbb{Z}\} \subset \mathbb{A}_{2\alpha}$ where

$$u_{kl} := \pi_\omega\big(W(f_k)W(g_l)\big)\,, \quad k,l \in \mathbb{Z}.^{**}$$

We can embed (left one) such elements in $\mathcal{H}_\omega = \bigoplus_{\mathbb{Z}} L^2(\mathbb{T}, \mathrm{d}m)$ by defining $e^{kl} := u_{kl}\xi_\omega$. We get

$$e^{kl}_n(z) = z^l \delta_{n,k}\,, \quad k,l \in \mathbb{Z}\,.$$

It is easily seen that

$$\langle e^{rs}, e^{RS} \rangle = \delta_{r,R} \oint z^{s-S} \frac{\mathrm{d}z}{2\pi \imath z} = \delta_{r,R}\delta_{s,S}\,, \qquad (6)$$

that is the e^{kl} form and orthonormal system.

Now we pass to the definition of the Fourier transform, which will be defined by looking at the left embedding of M into M_* as follows. For $a = \pi_\omega(A) \in M$.

$$x := \omega(\,\cdot\, a) =: L_a \in M_* = L^1(M,\omega) \equiv L^1(M),$$

** Here, $h_f(z)$ replaces one of the coordinates corresponding to $g(z) = z$. Notice that the new coordinate $h_f(z)$ is in general not smooth.

we put

$$\widehat{x}(k,l) := \omega\big(u^*_{kl}a\big)\,.$$

For general $x \in M_* = L^1(M) \equiv L^1_0(M)$, the formula assumes the form

$$\widehat{x}(k,l) = x(u^*_{kl}). \tag{7}$$

Al level of $L^2_0(M) \sim \mathcal{H}_\omega$, for $x = a\xi_\omega$ we obtain $\widehat{x}(k,l) = \langle x, e^{kl}\rangle$. as expected. Therefore, for generic elements $\xi \in \mathcal{H}_\omega$ we still get

$$\widehat{x}(k,l) = \langle x, e^{kl}\rangle, \quad k,l \in \mathbb{Z}.$$

The other possibility (essentially related to the right embedding) assumes the form

$$\widehat{x}(k,l) := \omega(au_{kl}), \quad k,l \in \mathbb{Z}. \tag{8}$$

At level of Hilbert spaces, for $\xi \in \mathcal{H}_\omega$ we get

$$\widehat{\xi}(k,l) := \langle x, E^{kl}\rangle, \quad k,l \in \mathbb{Z},$$

where the involved orthonormal bases is determined by $E^{kl} := J_\omega e^{kl}$, $k,l \in \mathbb{Z}$.

For both definitions, we first have the non-commutative generalisation of classical/tracial results (where # stands for each of the previously defined Fourier Transform, and for the corresponding left or right L^p-spaces):

(i) Riemann-Lebesgue Lemma: $(L^1(M,\omega))^{\#} \subsetneq c_o(\mathbb{Z}^2)$ with $x^{\#} = 0 \Rightarrow x = 0$, $\|x^{\#}\|_{c_o} \leqslant \|x\|_{L^1}$.

(ii) The Fourier transform is a unitary map between $L^2_{\#}(M)$ and $\ell^2(\mathbb{Z}^2)$:

$$\|x^{\#}\|_{\ell^2} = \|x\|_{L^2}, \quad x \in \mathcal{H}_\omega.$$

Consequently, The formula is well defined for all $L^p_{\#}(M)$, $1 \leqslant p \leqslant 2$.

(iii) Hausdorfff-Young Theorem: for $p \in [1,2]$ and $q \in [2,+\infty]$ its conjugate exponent, the

Fourier transform extends to a complete contraction

$$\mathcal{F}^{\#}_{p,q} : L^p_{\#}(M) \to \ell^q(\mathbb{Z}^2)\,.$$

In addition, for $p = 2$ $\mathcal{F}^{\#}_{2,2}$ provides a complete isometry when the involving Hilbert spaces are equipped with Pisier OH-structure.

(iv) Fejer and Abel Theorems: for $1 \leqslant p \leqslant 2$ and $x \in L^p_{\#}(M)$, we have for the limit in the L^p-norm,

$$\sum_{|k|,|l| \leqslant N} \left(1 - \frac{|k|}{N+1}\right)\left(1 - \frac{|l|}{N+1}\right) x^{\#}(k,l)\iota^{\#}_{\infty,p}(u_{k,l}) \to x.$$

$$\sum_{k,l \in \mathbb{Z}} r^{|k|+|l|}\; x^{\#}(k,l)\iota^{\#}_{\infty,p}(u_{k,l}) \to x,$$

when $N \to +\infty$, $r \uparrow 1$.

We can also define the corresponding Fourier anti-transforms

$$f \in \ell^1(\mathbb{Z}^2) \mapsto \begin{cases} \check{f} = & \sum_{k,l\in\mathbb{Z}} f(k,l)u_{kl} \\ \bar{f} = & \sum_{k,l\in\mathbb{Z}} f(k,l)u_{kl}^* \end{cases} \in M,$$

and prove the results corresponding to the (complete) boundedness for the restrictions to such maps to ℓ^p, $p \in [1,2]$, and some other natural ones.

modular spectral triples

When α is a **UL**-number defined before, we can construct a ''genuine'' modular spectral triple for $\mathbb{A}_{2\alpha}$.

For the corresponding twisted Dirac operators, we put

$$\mathsf{D}_n = \begin{pmatrix} 0 & \mathsf{L}_n \\ \mathsf{L}_n^* & 0 \end{pmatrix} := \begin{pmatrix} 0 & \left(\imath z\frac{\mathsf{d}}{\mathsf{d}z} - a_n I\right) \\ \left(-\imath z\frac{\mathsf{d}}{\mathsf{d}z} - a_n I\right) & 0 \end{pmatrix}$$

with

$$a_n := \mathsf{sign}(n) \sum_{l=1}^{|n|} \frac{1}{\Gamma_{l-\frac{1-\mathsf{sign}(n)}{2}}(f)}, \qquad n \in \mathbb{Z}.$$

For $\eta \in [0,1]$, we then define deformed Dirac Operators as

$$\begin{aligned} \mathsf{D}^{(\eta)} &= \bigoplus_{n\in\mathbb{Z}} \begin{pmatrix} 0 & M_{\delta_n^{\eta-1}} \mathsf{L}_n M_{\delta_n^{-\eta}} \\ M_{\delta_n^{-\eta}} \mathsf{L}_n^* M_{\delta_n^{\eta-1}} & 0 \end{pmatrix} \\ &= \bigoplus_{n\in\mathbb{Z}} \mathsf{D}_n^{(\eta)} = \begin{pmatrix} 0 & \Delta_\omega^{\eta-1} \mathsf{L} \Delta_\omega^{-\eta} \\ \Delta_\omega^{-\eta} \mathsf{L}^* \Delta_\omega^{\eta-1} & 0 \end{pmatrix}, \end{aligned}$$

with the associated twisted derivation

$$\mathcal{D}^{(\eta)} = \imath \begin{pmatrix} 0 & \Delta_\omega^{\eta-1} [\mathsf{L}, \cdot\,] \Delta_\omega^{-\eta} \\ \Delta_\omega^{-\eta} [\mathsf{L}^*, \cdot\,] \Delta_\omega^{\eta-1} & 0 \end{pmatrix}.$$

Here, Γ is the growth sequence previously defined, and

$$\delta_n(z) := \frac{\mathsf{d}m \circ f^n}{\mathsf{d}m}(z) = \frac{z(Df^n)(z)}{f^n(z)}, \qquad z \in \mathbb{T},\, n \in \mathbb{Z}$$

are nothing else the previous Radon-Nikodym derivatives describing the modular operator.

We can prove the following results, crucial to construct modular spectral triples:

(i) the $\mathsf{D}^{(\eta)}$ have compact resolvent, provided that $\Gamma_n(f) = o(\ln n)$;

(ii) the twisted commutators $\mathcal{D}^{(\eta)}$ admit in their domain a dense $*$-algebra $\mathbb{A}^o_{2\alpha} \subset \mathbb{A}_{2\alpha}$, stable under the entire functional calculus.

As an application of the above Fourier transforms, we can show that those "diagonalise" the deformed twisted operators $D^{(0)}$, $D^{(1)}$, and finally $D^{(1/2)}$, obtaining for $k, l, r, s \in \mathbb{Z}$,

$$\begin{aligned}\left\langle \Delta_\omega^{-1}\mathsf{L}e^{kl}, e^{rs}\right\rangle =&(\imath l - a_k)\widehat{1/\delta_k}(s-l)\delta_{k,r},\\ \left\langle \mathsf{L}\Delta_\omega^{-1}e^{kl}, e^{rs}\right\rangle =&(\imath s - a_k)\widehat{1/\delta_k}(s-l)\delta_{k,r},\\ \left\langle \Delta_\omega^{-1/2}\mathsf{L}\Delta_\omega^{-1/2}E^{kl}, E^{rs}\right\rangle =&-\big(\imath l\delta_{l,s} + a_{-k}\widehat{\delta_k}(l-s)\big)\delta_{k,r}.\end{aligned}$$

an intrinsic example of modular spectral triples

The models described in [1], Section 9, provide examples of modular spectral triples associated to non type II_1 representations. The situation of that paper provides spectral triples where the "smooth" operators constitute indeed a $*$-subalgebra. In the last situation however, the undeformed Dirac operator is non an intrinsic object but depends on the chosen state.

Indeed, consider the subset $\mathcal{C}_0 \subset C(\mathbb{T})$ defined by

$$\mathcal{C}_0 := \left\{ f \circ \hbar_{T^2} \mid f \in C^1(\mathbb{T}) \right\}.$$

For each finite subset $J \subset \mathbb{Z}$, consider the functions $f_k(m,n) := \widehat{F_k}(m)\delta_{n,0}$ with $F_k \in \mathcal{C}_0$, and $g_k(m,n) := \delta_{m,0}\delta_{n,k}$. We define $\mathcal{B}_0 \subset \mathcal{B}(\mathbb{Z}^2)$ as the set of all functions††

$$f := \sum_{k \in J} (f_k *_{2\alpha} g_k), \quad J \subset \mathbb{Z} \text{ finite}. \qquad (9)$$

††By definition, T is a "square-root of the diffeomorphism f, *i.e.* $T^2 = f$, and the right "coordinate" is $\hbar_f(z)$ instead of z.

We also define

$$\mathbb{A}_{2\alpha}^{oo} := \left\{ W(f) \mid f \in \mathcal{B}_0 \right\}.$$

One can see that $\mathbb{A}_{2\alpha}^{oo}$ is a unital $*$-algebra, which is dense in $\mathbb{A}_{2\alpha}$ by construction.[‡‡]

Indeed, fix such a diffeomorphism f of the circle, having rotation number 2α and prescribed ratio-set, and define the triple $\mathcal{T} := (\omega, \mathcal{L}, \mathcal{E})$ as follows. The state ω is constructed as previously described by the abelian dynamical system $(\mathbb{T}, f, \mathrm{d}\theta/2\pi)$, $\mathcal{L}$ is the intrinsic object in (1), and finally $\mathcal{E}$ is the subspace of $\mathcal{B}(\mathcal{H}_\omega)$ generated by $\{P_N \pi_\omega(\mathbb{A}_{2\alpha}^{oo}) P_N \mid N \in \mathbb{N}\}$ with P_N is the self-adjoint projection in $\mathcal{B}(\mathcal{H}_\omega)$ onto the subspace $\bigoplus_{|n| \leqslant N} L^2(\mathbb{T}, \mathrm{d}\theta/2\pi)$. Notice that, if we adjoin the multiples of the identity, $\mathcal{E} + \mathbb{C}I$ is an

[‡‡]We remark that $\mathbb{A}_{2\alpha}$ is in the domain of the twisted commutator $\mathcal{D}^{(\eta)}$. In addition, one can seen that, as the domain of $\mathcal{D}^{(\eta)}$, one can choose a dense $*$-algebra closed under the holomorphic functional calculus.

operator space. In addition, $\sup_N P_N = I$, and therefore $\mathcal{E}$ is "essential" in this sense.

We also note that, the Dirac operator deformed according to the modular structure associated to ω is given by

$$D_{\mathcal{T}} := \begin{pmatrix} 0 & \Delta_\omega^{-1} L \\ L^* \Delta_\omega^{-1} & 0 \end{pmatrix},$$

which has compact resolvent in all cases under consideration in the present situation.

The following result collects the computation in [1], Section 8, asserting that $\mathcal{T} := (\omega, \mathcal{L}, \mathcal{E})$ is a genuine spectral triple based on an essential operator space.

theorem
The operator system $\mathcal{E}$ is in the domain of the twisted derivation

$$\mathcal{D}^{(0)} = \imath \begin{pmatrix} 0 & \Delta_\omega^{-1}[L, \cdot\,] \\ [L^*, \cdot\,]\Delta_\omega^{-1} & 0 \end{pmatrix},$$

where L is the closed operator obtained by representing $\mathcal{L} = \partial_1 + \imath\partial_2$ on $\mathcal{H}_\omega$.

We now pass to discuss the possibility to define a kind of deformed Fredholm module which can arise from modular spectral triple. For this aim, define

$$\begin{aligned}\mathcal{D}_o :=\{\xi \bigoplus \eta \in \mathcal{H}_\omega \bigoplus \mathcal{H}_\omega \mid \xi_n, \eta_n \in AC(\mathbb{T}),\\ \xi'_n, \eta'_n \in L^2(\mathbb{T}, \mathrm{d}\theta/2\pi),\ n \in \mathbb{Z}\}\,.\end{aligned}$$

With a slightly abuse of notation, we put $D := D_\mathcal{T}$ with $DP^\perp_{\mathrm{Ker(D)}}$ acting on $P^\perp_{\mathrm{Ker(D)}}(\mathcal{H}_\omega \bigoplus \mathcal{H}_\omega)$. In such a situation, $D = |D|F = F|D|$ is non-singular, and $\Gamma := \begin{pmatrix} \Delta_\omega & 0 \\ 0 & I \end{pmatrix}$, and $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ for $a \in \mathcal{E}$, we define on $\mathcal{D}_o$,

$$\begin{aligned}[F,a]^{(o)}_\mathcal{T} := \big(F\Gamma A\Gamma^{-1} - \Gamma^{-1}A\Gamma F\big)\\ +\big(F|D|\Gamma A\Gamma^{-1}|D|^{-1} - |D|^{-1}\Gamma^{-1}A\Gamma|D|F\big)\,.\end{aligned}$$

We also put $d_\mathcal{T}(a) := \mathcal{D}^{(0)}(a)$.

theorem
For $a \in \mathcal{E}$, $[F,a]^{(o)}_{\mathcal{T}}$ uniquely extends to a bounded operator $[F,a]_{\mathcal{T}}$ satisfying

$$[F,a]_{\mathcal{T}} = |D_{\mathcal{T}}|^{-1} d_{\mathcal{T}}(a) + d_{\mathcal{T}}(a)|D_{\mathcal{T}}|^{-1}\,, \qquad (10)$$

Therefore, $[F,a]_{\mathcal{T}}$ is compact.

Notice that

- The domain of the deformed derivation $\mathcal{D}_o$ and the deformed Fredholm module can be enlarged to te operator system $\overline{\mathcal{E}}$, the closure of $\mathcal{E}$ under the "C^1-norm" $[\![a]\!]_{\mathcal{T}} := \|a\| + \|d_{\mathcal{T}}(a)\|$.

- Even in the usual case associated to the canonical trace of $\mathbb{A}_{2\alpha}$ when $\Gamma := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$, the our definition of the Fredholm module must be "deformed".

– We can provide a suitable general definition of a modular spectral triple, which gives rise to a deformed Fredholm module as above.

We end by briefly discussing a simple one dimensional example based on an slightly different one reported in "I. Forsyth, B. Mesland and A. Rennie: *Dense domains, symmetric operators and spectral triples*, New York J. Math. 20 (2014), 1001-1020".

example

We briefly discuss the 1-dimensional torus $\mathbb{T}$, whose (abelian) C^*-algebra is described by $C(\mathbb{T})$. In this situation, the Dirac operator $D := z\frac{\mathrm{d}}{\mathrm{d}z}$ is nothing else that the (positive) square root of the opposite of the laplacian. On a generator given by the multiplication operator $a := M_f$ for the function $f(z) = z^l$, and a function

$\xi(z) = \sum_k \xi_k z^k \in L^2\big(\mathbb{T}, \frac{\mathrm{d}z}{2\pi\imath z}\big)$ whose Fourier series has finite support, we have after disregarding the kernel of D consisting of the constant functions,

$$
\begin{aligned}
&[F_{\mathcal{T}}, a]\xi = (Fa - aF)\xi + (F|D|a|D|^{-1} - |D|^{-1}a|D|F)\xi \\
=&(Fa - |D|^{-1}aD)\xi + (Da|D|^{-1} - aF)\xi \\
=&\sum_{\{k|k,k+l\neq 0\}}\left[\left(\frac{k+l}{|k+l|} - \frac{k}{|k+l|}\right) + \left(\frac{k+l}{|k|} - \frac{k}{|k|}\right)\right] z^{k+l} \\
=&l\sum_{\{k|k,k+l\neq 0\}}\left(\frac{1}{|k+l|} + \frac{1}{|k|}\right) z^{k+l} \\
=&\big(|D|^{-1}[D,a] + [D,a]|D|^{-1}\big)\xi\,.
\end{aligned}
$$

The dense $*$-algebra $\mathcal{A}$ on which is based the (deformed) definition of the Fredholm module is given by

$$
\mathcal{A} := \overline{\mathrm{span}\{z^k \mid k \in \mathbb{Z}\}}^{\|\ \|_{\mathcal{T}}} = C^1(\mathbb{T})\,,
$$

where $\|\ \|_{\mathcal{T}}$ is the C^1-norm corresponding to that of the uniform convergence of functions together with their 1st derivative.

We also remark that, conversely to this simple model, Example 3.1 in the above cited paper cannot provide a Fredholm module (for both definitions, the usual undeformed and the deformed one given here) for this simple reason. On $C([0,2\pi])$ with one of the usual self-adjoint Dirac operators D_χ, $|\chi| = 1$ given by $-\imath\frac{\mathrm{d}}{\mathrm{d}\theta}$, with

$$\begin{aligned}\mathcal{D}_{D_\chi} =& \{f \in AC([0,2\pi]) \mid f(0)\\ &= \chi f(2\pi),\ f' \in L^2([0,2\pi])\}\end{aligned}$$

(*e.g.* Example in Section VIII.2 of Reed-Simon book), any dense $*$-subalgebra $\mathcal{A}$ of $C([0,2\pi])$ on which the commutator with one of the above Dirac operators provides bounded elements must contain a function f for which $f(0) \neq f(2\pi)$. For such a function, the asymptotic of the Fourier coefficients $\{\widehat{f}(k)\}_{k\in\mathbb{Z}}$ is at most $1/k$ or worst. Therefore, on one hand the deformed commutator $[F,a]_{\mathcal{T}}$, defined in any appropriate domain, is meaningless. On the other hand, $[F,a]$, again defined in any appropriate domain, cannot provide a compact operator as shown in the above paper.