Localisations and the Kasparov product in unbounded KK-theory

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KK-theory

- Let A be a separable C^*-algebra and let B, C be σ-unital C^*-algebras.

- **Definition** (Kasparov '80): A Kasparov A-B-module (A, E_B, F) is:
  - a ($\mathbb{Z}_2$-graded) Hilbert B-module E;
  - a $*$-homomorphism $A \to \mathrm{End}_B(E)$;
  - an (odd) operator $F \in \mathrm{End}_B(E)$ such that for all $a \in A$:

$$a(1 - F^2),\ a(F - F^*),\ [F, a] \text{ are compact operators.}$$

- The classes $[F] := [(A, E_B, F)]$ modulo the "homotopy equivalence" relation form an abelian group $KK(A, B)$.
  If $a(F_1 - F_2)$ is compact for all $a \in A$, then $[F_1] = [F_2]$.

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Typical example: elliptic 0th-order ΨDO F

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  └ "locally compact" perturbation

Kasparov product

- The Kasparov product is an associative bilinear pairing

$$KK(A,B) \times KK(B,C) \to KK(A,C).$$

- **Theorem** (Connes-Skandalis '84) Consider Kasparov modules $(A, (E_1)_B, F_1)$, $(B, (E_2)_C, F_2)$, and (A, E_C, F) with $E = E_1 \otimes_B E_2$. Assume the following conditions are satisfied:

  connection: for all $\psi \in E_1$, the graded commutator
  $$\left[\begin{pmatrix} F & 0 \\ 0 & F_2 \end{pmatrix}, \begin{pmatrix} 0 & T_\psi \\ T_\psi^* & 0 \end{pmatrix} \right] \text{ is compact on } E \oplus E_2,$$
  where $T_\psi \colon E_2 \to E$, $\eta \mapsto \psi \otimes \eta$;

  positivity: there exists $0 \leq \kappa < 2$ such that for all $a \in A$:
  $$a^*[F_1 \otimes 1, F]a \geq -\kappa a^* a \quad \text{modulo compact operators.}$$

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  Then $[F] = [F_1] \otimes_B [F_2] \in KK(A,C)$. Moreover such F always exists!

Unbounded KK-theory

- **Definition** (Baaj-Julg '83, Hilsum '10)
  A half-closed module $(\mathcal{A}, E_B, D)$ is:
  - a ($\mathbb{Z}_2$-graded) Hilbert B-module E;
  - a $*$-homomorphism $A \to \mathrm{End}_B(E)$;
  - an (odd) regular symmetric operator $\mathcal{D}$ on E such that for all $a \in A$: $a(1 + \mathcal{D}^*\mathcal{D})^{-1}$ is compact;
  - a dense $*$-subalgebra $\mathcal{A} \subset A$ such that for all $a \in \mathcal{A}$: $a \cdot \mathrm{Dom}\, \mathcal{D}^* \subset \mathrm{Dom}\, \mathcal{D}$ and $[\mathcal{D}, a]$ is bounded.

- If $\mathcal{D} = \mathcal{D}^*$, then $(\mathcal{A}, E_B, \mathcal{D})$ is an *unbounded Kasparov module.*

- **Theorem** (Baaj-Julg '83, Hilsum '10)
  Consider the bounded transform $F_{\mathcal{D}} := \mathcal{D}(1 + \mathcal{D}^*\mathcal{D})^{-1/2}$.
  Then $(A, E_B, F_{\mathcal{D}})$ is a Kasparov module, and the map $(\mathcal{A}, E_B, \mathcal{D}) \mapsto [F_{\mathcal{D}}]$ is surjective onto $KK(A, B)$.

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Typical example: elliptic 1st-order diff op. Eg: $(C_c^\infty(0,1), L^2(0,1), D_{(0,1)} : i\partial_x)$

- If $\mathcal{D} = \mathcal{D}^*$, then $(\mathcal{A}, E_B, \mathcal{D})$ is an *unbounded Kasparov module.*

  Eg: $(C_c^\infty(\mathbb{R}), L^2(\mathbb{R}), D_{\mathbb{R}} = i\partial_x)$ $[D_{\mathbb{R}}] \simeq [D_{(0,1)}]$
- **Theorem** (Baaj-Julg '83, Hilsum '10)
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  We define $[D] := [F_D]$.

Kucerovsky's Theorem

- **Theorem** (Kucerovsky '97) Consider *unbounded* Kasparov modules $(\mathcal{A}, (E_1)_B, \mathcal{D}_1)$, $(\mathcal{B}, (E_2)_C, \mathcal{D}_2)$, and $(\mathcal{A}, E_C, \mathcal{D})$ with $E = E_1 \otimes_B E_2$. Assume the following conditions are satisfied:

  connection: for ψ dense in E_1, the graded commutator
  $$\left[\begin{pmatrix} \mathcal{D} & 0 \\ 0 & \mathcal{D}_2 \end{pmatrix}, \begin{pmatrix} 0 & T_\psi \\ T_\psi^* & 0 \end{pmatrix}\right] \text{ is bounded on } E \oplus E_2;$$
  positivity: we have $\operatorname{Dom}\mathcal{D} \subset \operatorname{Dom}\mathcal{D}_1 \otimes 1$, and there exists $c \geq 0$ such that for all $\xi \in \operatorname{Dom}\mathcal{D}$:
  $$\langle(\mathcal{D}_1 \otimes 1)\xi \,|\, \mathcal{D}\xi\rangle + \langle \mathcal{D}\xi \,|\, (\mathcal{D}_1 \otimes 1)\xi\rangle \geq -c\langle\xi|\xi\rangle.$$

Then $[\mathcal{D}] = [\mathcal{D}_1] \otimes_B [\mathcal{D}_2] \in KK(A, C)$.

Recall: CS positivity: $a^*[F, \otimes 1, F]a \geq 0$ "mod neg-order ΨDO's"

Kucerovsky: "$[D, \otimes 1, D] \geq 0$ mod 0th-order ops"

The positivity condition

- There is room for improvement in Kucerovsky's positivity condition:
  (I) it depends on the subprincipal symbol of $[\mathcal{D}, \mathcal{D}_1 \otimes 1]$;
  (II) it is global instead of local.
- **Remark:** localising the positivity condition also allows us to consider non-selfadjoint operators (i.e., half-closed modules).
- **Example:** Consider the manifold $M := (0,1) \times (0,1)$, the C^*-algebras $A := C_0(M)$, $B = C_0(0,1)$, and $C = \mathbb{C}$, and the operators

$$\mathcal{D}_1 := i\partial_y + \sin\left(\frac{1}{x}\right), \quad \text{on } E_1 := C_0\big((0,1), L^2(0,1)\big),$$
$$\mathcal{D}_2 := i\partial_x, \quad \text{on } E_2 := L^2(0,1).$$

The operator $\mathcal{D} := \mathcal{D}_1\sigma_1 + \mathcal{D}_2\sigma_2$ represents the Kasparov product $[\mathcal{D}_1] \otimes_B [\mathcal{D}_2]$, *but Kucerovsky's positivity condition fails!*

The positivity condition

$K \subset (0,1)$ cpt

$D_{(0,1)}|_K = D_{\mathbb{R}}|_K$

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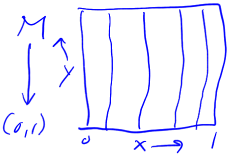

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$$[D_1, D_2] = -i\partial_x(\sin(\tfrac{1}{x})) = \frac{i}{x^2}\cos(\tfrac{1}{x})$$

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Positivity ‘modulo first-order’

Theorem (vdD’20)

Consider unbounded Kasparov modules $(A,(E_1)_B,\mathcal{D}_1)$, $(B,(E_2)_C,\mathcal{D}_2)$, and $(A,E_C,\mathcal{D})$ with $E=E_1\otimes_B E_2$. Assume the following conditions are satisfied:

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"$[\mathcal{D}_1\otimes 1,\mathcal{D}]\geq 0$ mod 1^{st}-order op's"

The localised representative

- Let $(\mathcal{A}, E_B, \mathcal{D})$ be a half-closed module, for which the representation $A \to \mathrm{End}_B(E)$ is essential.
- **Assumption:** $\mathcal{A} \subset A$ contains an (even) approximate unit $\{u_n\}_{n\in\mathbb{N}}$ for A which is almost idempotent: $u_{n+1}u_n = u_n$ for all $n \in \mathbb{N}$.
- We obtain a *"partition of unity"* $\{\chi_k{}^2\}_{k\in\mathbb{N}}$:

$$\chi_0 := u_0{}^{1/2}, \qquad \chi_k := (u_k - u_{k-1})^{1/2}, \quad k > 1.$$

- **Lemma:** The 'localised operator' $\mathcal{D}_k := u_{k+2}\mathcal{D}u_{k+2}$ is regular and self-adjoint.
- **Definition:** For any sequence $\{\alpha_k\}_{k\in\mathbb{N}} \subset (0,\infty)$, the localised representative of $(\mathcal{A}, E_B, \mathcal{D})$ is

$$\widetilde{F}_{\mathcal{D}}(\alpha) := \sum_{k=0}^{\infty} \chi_k F_{\alpha_k \mathcal{D}_k} \chi_k .$$

- **Proposition** (vdD'20): $[\widetilde{F}_{\mathcal{D}}(\alpha)] = [F_{\mathcal{D}}] \in KK(A,B)$.

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$\chi_k (F_{D_k} - F_D) = cpt$

$\chi_k (F_{D_k} - F_{\alpha_k D_k}) = cpt$

$\Rightarrow u_n (\widetilde{F}_D(\alpha) - F_D) = cpt$

- **Proposition (vdD'20):** $[\widetilde{F}_{\mathcal{D}}(\alpha)] = [F_{\mathcal{D}}] \in KK(A, B)$.

Local positivity condition

- **Assumptions:** Consider half-closed modules $(\mathcal{A}, (E_1)_B, \mathcal{D}_1)$ (with essential representation), $(\mathcal{A}, (E_2)_C, \mathcal{D}_2)$, and $(\mathcal{A}, E_C, \mathcal{D})$, with $E := E_1 \otimes_B E_2$. We assume that $\mathcal{A} \subset A$ contains an (even) almost idempotent approximate unit $\{u_n\}$ for A.

- **Definition:** the strong local positivity condition requires for each $n \in \mathbb{N}$ that we have $u_n \cdot \operatorname{Dom} \mathcal{D} \subset \operatorname{Dom} \mathcal{D}_1 \otimes 1$, and there exist $\nu_n > 0$ and $c_n \geq 0$ such that for all $\xi \in \operatorname{Dom} \mathcal{D}$:

$$\langle (\mathcal{D}_1 \otimes 1) u_n \xi \mid \mathcal{D} u_n \xi \rangle + \langle \mathcal{D} u_n \xi \mid (\mathcal{D}_1 \otimes 1) u_n \xi \rangle \\ \geq \nu_n \langle (\mathcal{D}_1 \otimes 1) u_n \xi \mid (\mathcal{D}_1 \otimes 1) u_n \xi \rangle - c \langle u_n \xi \mid (1 + \mathcal{D}^* \mathcal{D})^{-1/2} u_n \xi \rangle.$$

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Kucerovsky's Theorem localised

Theorem (vdD'20)

Consider half-closed modules $(\mathcal{A}, (E_1)_B, \mathcal{D}_1)$ (with essential representation), $(\mathcal{A}, (E_2)_C, \mathcal{D}_2)$, and $(\mathcal{A}, E_C, \mathcal{D})$, with $E := E_1 \otimes_B E_2$. Assume that (Kucerovsky's) connection condition is satisfied, and that $\mathcal{A} \subset A$ contains an (even) almost idempotent approximate unit $\{u_n\}$ for A, such that the strong local positivity condition is satisfied. Then $[\mathcal{D}] = [\mathcal{D}_1] \otimes_B [\mathcal{D}_2] \in KK(A, C)$.

Proof Step 1 strong loc pos for D, D_1 $\Rightarrow$ strong loc pos for $D_k, D_{1,k}$ $\Rightarrow$ CS positivity "locally":

Given $0<\kappa<2$, $\forall k \in \mathbb{N}$ $\exists \alpha_k > 0$ s.t.

$$\chi_k [F_{D_k}, F_{\alpha_k D_{1,k}} \otimes 1] \chi_k \geq -\kappa \chi_k^2 \mod \text{cpts}$$

Step 2: CS positivity globally for F_D and $\tilde{F}_{D_1}(\alpha)$:

$$u_n [F_D, \tilde{F}_{D_1}(\alpha) \otimes 1] u_n = \sum_k u_n \chi_k [F_{D_k}, F_{\alpha_k D_{1,k}} \otimes 1] \chi_k u_n \mod \text{cpts}$$

$$\geq -\kappa u_n^2 \mod \text{cpts}. \quad \square$$

The constructive approach

- **Assumption:** Given two half-closed modules $(\mathcal{A}, (E_1)_B, \mathcal{D}_1)$ (with essential representation) and $(\mathcal{B}, (E_2)_C, \mathcal{D}_2)$, write $\mathcal{S} := \mathcal{D}_1 \otimes 1$ on $E := E_1 \otimes_B E_2$. Consider an (odd) symmetric operator $\mathcal{T}$ on E, and write $\mathcal{D} := \overline{\mathcal{S} + \mathcal{T}}$. We assume:

(A1) $\mathcal{D}$ yields a half-closed module $(\mathcal{A}, E_C, \mathcal{D})$;

(A2) for ψ dense in $\mathcal{A} \cdot \operatorname{Dom} \mathcal{D}_1$, the graded commutator

$$\left[\begin{pmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{D}_2 \end{pmatrix}, \begin{pmatrix} 0 & T_\psi \\ T_\psi^* & 0 \end{pmatrix} \right] \text{ is bounded on } E \oplus E_2;$$

(A3) $\mathcal{A} \subset A$ contains an (even) almost idempotent approximate unit $\{u_n\}$ for A, such that $u_n \cdot \operatorname{Dom} \mathcal{D} \subset \operatorname{Dom} \mathcal{S} \cap \operatorname{Dom} \mathcal{T}$.

- **Remark:** (A2) ensures the connection condition is satisfied.

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$S T_\psi = T_{0,\psi}$ = bdd on E_2 $\forall \psi \in \operatorname{dom} \mathcal{D}_1$

The constructive approach

Theorem (vdD'20)

Given two half-closed modules $(\mathcal{A}, (E_1)_B, \mathcal{D}_1)$ (with essential representation) and $(\mathcal{B}, (E_2)_C, \mathcal{D}_2)$, write $\mathcal{S} := \mathcal{D}_1 \otimes 1$ on $E := E_1 \otimes_B E_2$. Consider an (odd) symmetric operator $\mathcal{T}$ on E, and write $\mathcal{D} := \overline{\mathcal{S} + \mathcal{T}}$. Suppose the assumptions (A1)-(A3) are satisfied. We assume there exists a core $\mathcal{F} \subset \operatorname{Dom} \mathcal{D}$ such that for all $n \in \mathbb{N}$:

- *$u_n \cdot \mathcal{F} \subset \operatorname{Dom} \mathcal{S}\mathcal{T} \cap \operatorname{Dom} \mathcal{T}\mathcal{S}$; and*
- *there exist $c_n \geq 0$ such that for all $\eta \in \mathcal{F}$:*

$$\big\|[\mathcal{S}, \mathcal{T}]u_n\eta\big\| \leq c_n\big\|(1 + \mathcal{D}^*\mathcal{D})^{1/2}u_n\eta\big\|.$$

Then $[\mathcal{D}] = [\mathcal{D}_1] \otimes_B [\mathcal{D}_2] \in KK(A, C)$.

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↪ "[S,T] is 1st-order"

Then $[\mathcal{D}] = [\mathcal{D}_1] \otimes_B [\mathcal{D}_2] \in KK(A, C)$.

Proof idea: $[D, S] = 2S^2 + [S,T] \geq 2S^2 - c_n (1+D^*D)^{1/2}$

we can choose $\nu_n = 2$ in the strong loc pos condition

References

Baaj-Julg'83 Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules hilbertiens, C. R. Acad. Sci. Paris Sér. I Math. 296 (1983), 875–878.

Connes-Skandalis'84 The longitudinal index theorem for foliations, Publ. Res. Inst. Math. Sci. 20 (1984), 1139–1183.

Hilsum'10 Bordism invariance in KK-theory, Math. Scand. 107 (2010), 73–89.

Kasparov'80 The operator K-functor and extensions of C^*-algebras, Izv. Akad. Nauk SSSR 44 (1980), 571–636.

Kucerovsky'97 The KK-product of unbounded modules, K-Theory 11 (1997), 17–34.

vdD Localisations of half-closed modules and the unbounded Kasparov product. arXiv:2006.10616