Positive Line Bundles Over the Irreducible Quantum Flag Manifolds

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Drinfeld–Jimbo quantum groups

Let $q \in \mathbb{R}\backslash\{-1,0,1\}$. Every finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ admits a q-deformation of its universal enveloping algebra, $U_q(\mathfrak{g})$:

- rank($\mathfrak{g}$) = r
- $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$, with $\Delta \subset \mathfrak{h}^*$ corresponding root system
- $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ choice simple roots
- $(\,\cdot\,,\,\cdot\,) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ symmetric bilinear form induced by Killing form of $\mathfrak{g}$, normalised so that $(\alpha_i, \alpha_i) = 2$ for any shortest α_i
- Cartan matrix $A = (a_{ij})_{i,j=1,\ldots,r}$ where $a_{ij} = (\alpha_i^\vee, \alpha_j)$ for coroot $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$

Quantised enveloping algebra

Let $q_i = q^{(\alpha_i,\alpha_i)/2}$. The quantised enveloping algebra $U_q(\mathfrak{g})$ is generated by E_i, F_i, K_i, K_i^{-1} for $i = 1,\ldots,r$ subject to relations

$$K_iE_j = q_i^{a_{ij}}E_jK_i, \quad K_iF_j = q_i^{-a_{ij}}F_jK_i, \quad K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

and the quantum Serre relations.

Hopf algebra structure of $U_q(\mathfrak{g})$

The quantised enveloping algebra $U_q(\mathfrak{g})$ admits a Hopf algebra structure:

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_0 \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

$$S(E_i) = -E_iK_i^{-1}, \quad S(F_i) = -K_iF_i, \quad S(K_i) = K_i^{-1},$$

$$\epsilon(E_i) = \epsilon(F_i) = 0, \quad \epsilon(K_i) = 1.$$

The *compact real form* of $U_q(\mathfrak{g})$ is given by the Hopf *-algebra structure

$$K_i^* = K_i, \quad E_i^* = K_iF_i, \quad F_i^* = E_iK_i^{-1}.$$

Type-1 representations

- $\mathscr{P}$ the weight lattice of $\mathfrak{g}$
- $\mathscr{P}^+$ dominant integral weights

For every $\mu \in \mathscr{P}^+$ there exists an irreducible finite-dimensional (left) $\mathscr{U}_q(\mathfrak{g})$-module V_μ uniquely defined by the existence of a *highest weight vector* $v_\mu \in V_\mu$ such that

$$E_i \triangleright v_\mu = 0, \quad K_i \triangleright v_\mu = q^{(\mu,\alpha_i)} v_\mu, \qquad \text{for } i = 1, \ldots, r\,.$$

The vector v_μ is unique up to scalar multiple. A finite direct sum of such modules is called a *type-1 representation*.

A vector $v \in V_\mu$ is called a *weight vector with weight* $\mathrm{wt}(v) \in \mathscr{P}$ if

$$K_i \triangleright v = q^{(\mathrm{wt}(v),\alpha_i)} v, \quad \text{for all } i = 1, \ldots, r\,.$$

Quantum coordinate algebra

Let V be a finite-dimensional left $U_q(\mathfrak{g})$-module and let V^* be its $\mathbb{C}$-linear dual with right $U_q(\mathfrak{g})$-module structure. Given $v \in V$, $f \in V^*$, define

$$c^V_{f,v} : U_q(\mathfrak{g}) \to \mathbb{C}, \qquad X \mapsto f(X \triangleright v)\,.$$

The coordinate ring of V is then given by

$$C(V) := \mathrm{span}_{\mathbb{C}}\{c^V_{f,v} \mid v \in V, f \in V^*\} \subset U_q(\mathfrak{g})^*\,.$$

Let G be the compact, connected, simply-connected simple Lie group with $\mathfrak{g}$ as its complexified Lie algebra. The *quantum coordinate algebra* of G is the Hopf subalgebra of $U_q(\mathfrak{g})^\circ$ given by

$$\mathcal{O}_q(G) := \bigoplus_{\mu \in \mathcal{P}^+} C(V_\mu)\,.$$

Quantum Levi subalgebra

Let $S \subset \{1, \ldots, r\}$ and consider the subset $\{\alpha_i\}_{i \in S}$ of simple roots.

Then there is a Hopf *-algebra embedding of the *quantum Levi subalgebra* $U_q(\mathfrak{l}_S)$

$$\iota_S : U_q(\mathfrak{l}_S) := \langle K_i, E_j, F_j \mid i = 1, \ldots, r; j \in S \rangle \hookrightarrow U_q(\mathfrak{g}) \,.$$

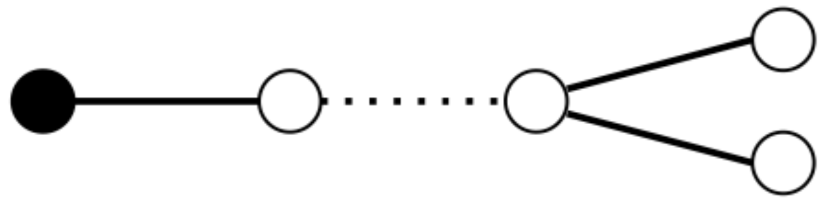
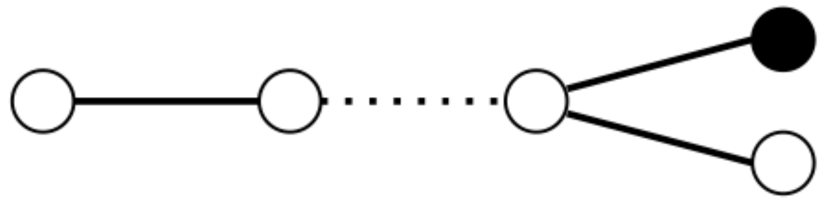
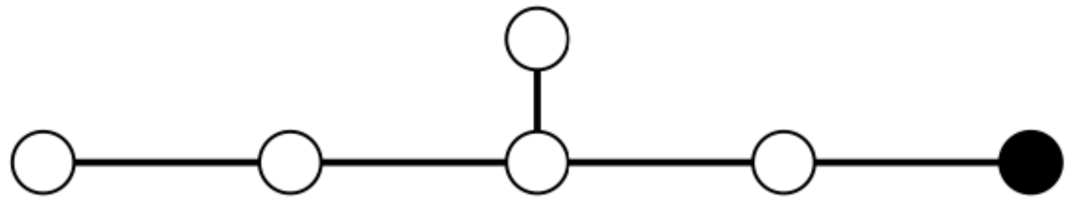
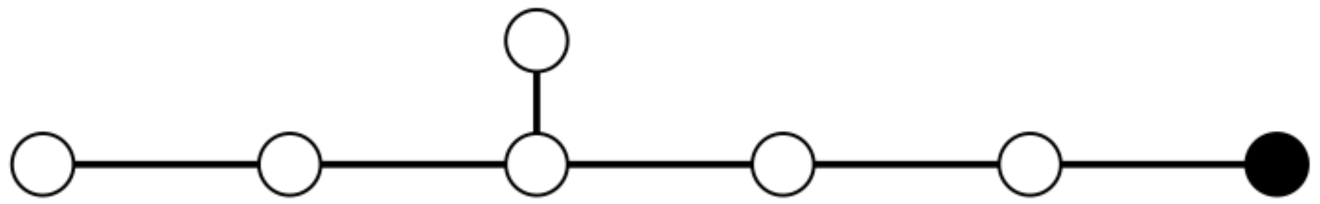
Quantum flag manifold $\mathcal{O}_q(G/L_S)$

The dual pairing $\langle \cdot\,,\cdot \rangle$ between $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$ gives a left action $\triangleright$ of $U_q(\mathfrak{l}_S)$ on $\mathcal{O}_q(G)$. The *quantum flag manifold* $\mathcal{O}_q(G/L_S)$ is then given by the space of space of invariant elements,

$$\mathcal{O}_q(G/L_S) := {}^{U_q(\mathfrak{l}_S)}\mathcal{O}_q(G) = \{a \in \mathcal{O}_q(G) \mid X \triangleright a = \epsilon(X)a\}\,.$$

When $S = \{1,\ldots,r\}\backslash\{s\}$ where α_s has coefficient 1 in the expansion of the highest root of $\mathfrak{g}$, the quantum flag manifold $\mathcal{O}_q(G/L_S)$ is called *irreducible*.

Irreducible quantum flag manifolds

A_n		$\mathcal{O}_q(\mathrm{Gr}_{s,n+1})$	quantum Grassmannian
B_n		$\mathcal{O}_q(\mathbf{Q}_{2n+1})$	odd quantum quadric
C_n		$\mathcal{O}_q(\mathbf{L}_n)$	quantum Lagrangian Grassmannian
D_n		$\mathcal{O}_q(\mathbf{Q}_{2n})$	even quantum quadric
D_n		$\mathcal{O}_q(\mathbf{S}_n)$	quantum spinor variety
E_6		$\mathcal{O}_q(\mathbb{OP}^2)$	quantum Caley plane
E_7		$\mathcal{O}_q(\mathrm{F})$	quantum Freudenthal variety

Flag manifolds are great!

Classical flag manifolds have an extremely rich geometric structure. For example:

- They are complex manifolds.
- In fact, they are Kähler manifolds.
- In fact, they can be equivalently described as compact homogeneous Kähler manifolds (corresponding to a compact connected semisimple Lie group).
- An irreducible flag manifold is moreover a symmetric manifold. (These are some of the best manifolds you could ask for!)
- The Borel–Weil theorem for line bundles over flag manifolds is a starting point for geometric representation theory.
- The study of their cohomology, Schubert Calculus, has fundamental connections to combinatorics and integrable systems.
- As a resident of the Czech Republic, I'd be remiss not to mention parabolic geometry...

Question: Do quantum flag manifolds admit a noncommutative geometry in common with their classical counterparts?

Differential calculus over a *-algebra

Let B be a *-algebra. A *differential calculus* $(\Omega^\bullet = \Omega_{k \in \mathbb{N}} \Omega^k, \mathrm{d})$ *over* B is a differential graded algebra generated by elements of the form $a, \mathrm{d}b$ for $a, b \in \Omega^0 \cong B$.

A *differential *-calculus over B* is a differential calculus such that the *-map of B extends to a conjugate-linear involution $* : \Omega^\bullet \to \Omega^\bullet$ s.t.

- $\mathrm{d}(\omega^*) = (\mathrm{d}\omega)^*, \quad \omega \in \Omega^\bullet$
- $(\omega \wedge \nu)^* = (-1)^{kl} \nu^* \wedge \omega^*, \qquad \omega \in \Omega^k, \nu \in \Omega^l$.

The *total degree* of $\Omega^\bullet$ is the least integer m such that $\Omega^k = 0$ for every $k > m$.

Let A be a Hopf *-algebra and suppose that B is a left A-comodule algebra. We say $(\Omega^\bullet, \mathrm{d})$ is *left (A-)covariant* if the coaction $\Delta_L : B \to A \otimes B$ extends to a left coaction $\Delta_L : \Omega^\bullet \to A \otimes \Omega^\bullet$.

Complex structures

An *almost complex structure* $\Omega^{(\bullet,\bullet)}$ for a differential $*$-calculus $\Omega^\bullet$ is an $\mathbb{N}^2$-grading $\bigoplus_{(a,b)\in\mathbb{N}\times\mathbb{N}} \Omega^{(a,b)} \cong \Omega^\bullet$ such that, for every $(a,b) \in \mathbb{N}^2$,

- $\Omega^k = \bigoplus_{a+b=k} \Omega^{(a,b)}$,
- $(\Omega^{(a,b)})^* = \Omega^{(b,a)}$.

A *complex structure* is an almost complex structure satisfying

- $d\Omega^{(a,b)} \subset \Omega^{(a+1,b)} \oplus \Omega^{(a,b+1)}$, for every $(a,b) \in \mathbb{N}^2$.

E. Beggs, S. Paul Smith / Journal of Geometry and Physics 72 (2013)

Masoud Khalkhali ; Giovanni Landi ; Walter Daniël van Suijlekom
International Mathematics Research Notices (Volume: 2011 , Issue: 4, 2011)

If B is a left A-comodule algebra then a complex structure is covariant if $\Omega^\bullet$ is covariant and the $\mathbb{N}^2$-decomposition is in the category of left A-comodules.

Dolbeault Double Complex

Let $\Omega^{(\bullet,\bullet)}$ be an almost complex structure and define projections

$$\mathrm{proj}_{\Omega^{(a+1,b)}} : \Omega^{a+b+1} \to \Omega^{a+1,b}, \qquad \mathrm{proj}_{\Omega^{(a,b+1)}} : \Omega^{a+b+1} \to \Omega^{a,b+1},$$

and let ∂ and $\overline{\partial}$ be defined by

$$\partial|_{\Omega^{(a,b)}} = \mathrm{proj}_{\Omega^{(a+1,b)}} \circ \mathrm{d}, \qquad \overline{\partial}|_{\Omega^{(a,b)}} = \mathrm{proj}_{\Omega^{(a,b+1)}} \circ \mathrm{d}\,.$$

If $\Omega^\bullet$ has a complex structure then

$$\mathrm{d} = \partial + \overline{\partial}, \qquad \overline{\partial} \circ \partial = -\,\partial \circ \overline{\partial}, \qquad \partial^2 = \overline{\partial}^2 = 0.$$

The double complex $\left(\bigoplus_{(a,b)\in\mathbb{N}^2} \Omega^{(a,b)}, \partial, \overline{\partial} \right)$ is called the *Dolbeault double complex* of $\Omega^\bullet$.

Note that $\partial, \overline{\partial}$ satisfy the graded Leibniz rule.

Example: Quantum projective plane

$$
\begin{tikzcd}
\Omega^4 \\
\Omega^3 \arrow[u] \\
\Omega^2 \arrow[u] \\
\Omega^1 \arrow[u, "d"'] \\
\Omega^0 \arrow[u]
\end{tikzcd}
\qquad d = \partial + \bar{\partial}
$$

$$
\begin{tikzcd}
 & & \Omega^{(2,2)} & & \\
 & \Omega^{(2,1)} \arrow[ur] & & \Omega^{(1,2)} \arrow[ul] & \\
\Omega^{(2,0)} \arrow[ur] & & \Omega^{(1,1)} \arrow[ul] \arrow[ur] & & \Omega^{(0,2)} \arrow[ul] \\
 & \Omega^{(1,0)} \arrow[ul, "\partial"] \arrow[ur] & & \Omega^{(0,1)} \arrow[ul] \arrow[ur, "\bar{\partial}"'] & \\
 & & \Omega^{(0,0)} \arrow[ul, "\partial"] \arrow[ur, "\bar{\partial}"'] & &
\end{tikzcd}
$$

$$\Omega^{(0,0)} \cong \mathcal{O}_q(\mathbb{CP}^2)$$

Kähler structures

Let $\Omega^{(\bullet,\bullet)}$ be a complex structure for a differential *-calculus over a *-algebra B. If $\Omega^\bullet$ has total degree $2n$ and there exists a central closed real form $\kappa \in \Omega^{(1,1)}$ which gives isomorphisms

$$L^{n-k} : \Omega^k \to \Omega^{2n-k}, \qquad k = 0,\ldots,n-1$$

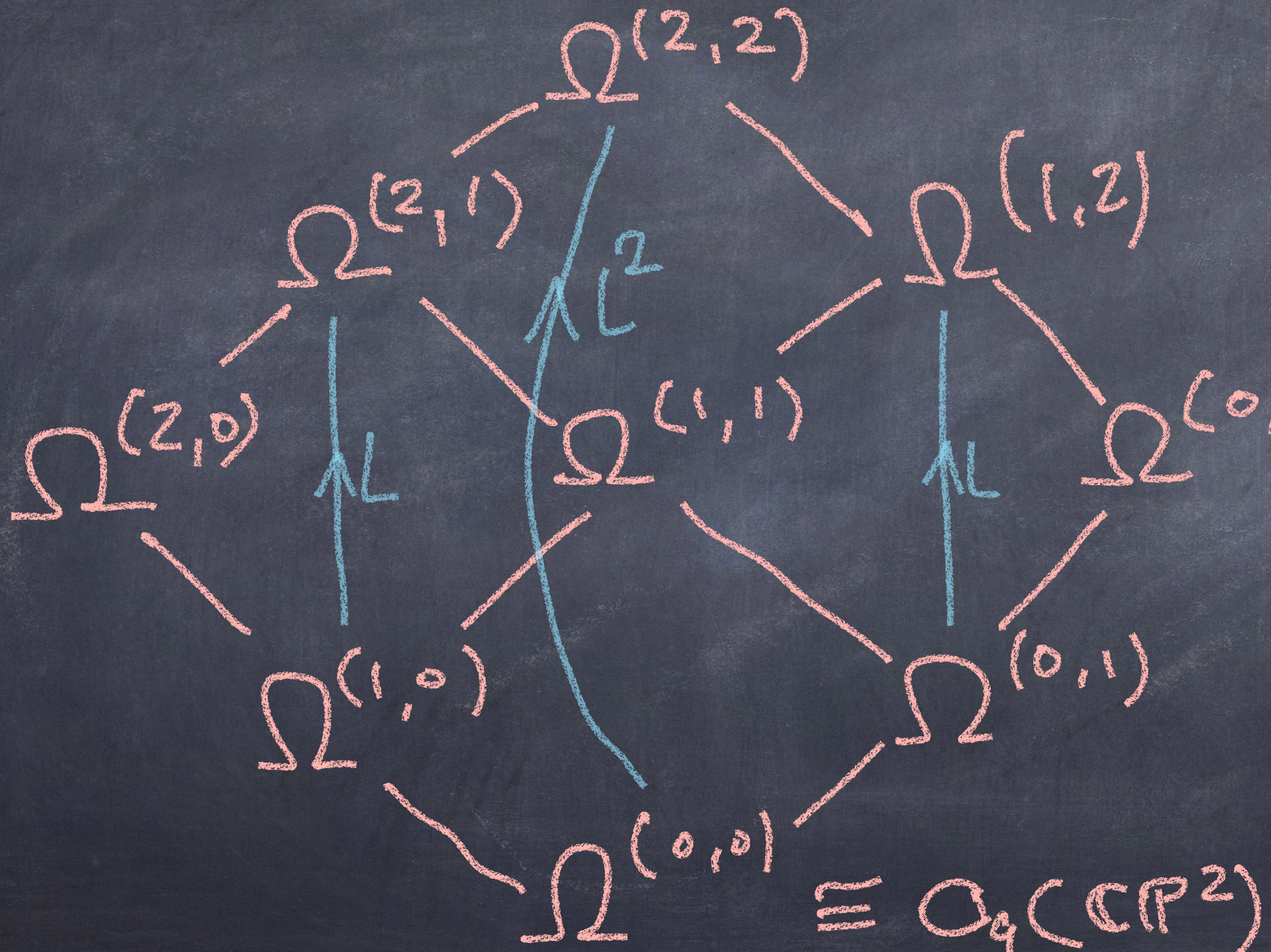
where $L : \Omega^\bullet \to \Omega^\bullet, \omega \to \kappa \wedge \omega$ is the *Lefschetz operator,* then we say that the pair $(\Omega^{(\bullet,\bullet)}, \kappa)$ is a *Kähler structure*. There exists a canonical *Hodge map* $*_\kappa : \Omega^\bullet \to \Omega^\bullet$ in analogy with the classical case.

A Kähler structure is *positive definite* if the associated *Kähler metric*

$$g_\kappa(\omega,\nu) := \begin{cases} *_\kappa(\omega \wedge *_\kappa(\nu^*)), & k = l, \\ 0, & k \neq l\,. \end{cases}$$

is positive definite (ie. for every nonzero $\omega \in \Omega^\bullet$, $g_\kappa(\omega,\omega) = b^*b$ for some non-zero $b \in B$).

Example: Quantum projective plane



A long list of quantum miracles occurs*…

The existence of a Kähler structure implies many nice properties. For example, just looking at cohomological implications:

- Hodge decomposition of the de Rham complex ($\mathrm{d}, \partial, \overline{\partial}$ versions)
- Lefschetz identities $\Longrightarrow$ Hard Lefschetz theorem
- Kähler identities $\Longrightarrow$ Dolbeault cohomology refines de Rham cohomology: $H^k_{\mathrm{d}} \cong \oplus_{a+b=k} H^{(a,b)}_{\partial} \cong \oplus_{a+b=k} H^{(a,b)}_{\overline{\partial}}$
- Serre duality
- Kodaira vanishing for positive line bundles
- …

*to paraphrase Bourguignon

Heckenberger–Kolb calculi

Theorem *Over any irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, there exists a unique finite-dimensional left $\mathcal{O}_q(G)$-covariant differential $*$-calculus*

$$\Omega_q^\bullet(G/L_S) \in {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{mod}$$

which is of classical dimension, that is to say, satisfying

$$\dim \Phi\big(\Omega_q^k(G/L_S)\big) = \binom{2M}{k}, \qquad \text{for all } k = 0, \ldots, 2M,$$

where M is the complex dimension of the corresponding classical manifold.

I. HECKENBERGER AND S. KOLB *Proc. London Math. Soc.* (3) 89 (2004)

I. Heckenberger, S. Kolb / Journal of Algebra 305 (2006)

	$\mathcal{O}_q(G/L_S)$	$M := \dim\big(\Omega^{(1,0)}\big)$
quantum Grassmannian	$\mathcal{O}_q(\mathrm{Gr}_{s,n+1})$	$s(n-s+1)$
odd quantum quadric	$\mathcal{O}_q(\mathbf{Q}_{2n+1})$	$2n-1$
quantum Lagrangian Grassmannian	$\mathcal{O}_q(\mathbf{L}_n)$	$\frac{n(n+1)}{2}$
even quantum quadric	$\mathcal{O}_q(\mathbf{Q}_{2n})$	$2(n-1)$
quantum spinor variety	$\mathcal{O}_q(\mathbf{S}_n)$	$\frac{n(n-1)}{2}$
quantum Cayley plane	$\mathcal{O}_q(\mathbb{OP}^2)$	16
quantum Freudentahl variety	$\mathcal{O}_q(\mathrm{F})$	27

Heckenberger–Kolb calculi are Kähler

(Matassa) The Heckenberger–Kolb calculus of an irreducible quantum flag manifold admits a unique covariant Kähler structure.

Consequently, we have all the properties that come along with a Kähler structure, in particular the important results on cohomology we saw before:

- Hodge decomposition
- Hard Lefschetz theorem
- Serre duality
- Dolbeault refinement of de Rham cohomology:
  $H^k_{\mathrm{d}} \cong \bigoplus_{a+b=k} H^{(a,b)}_{\partial} \cong \bigoplus_{a+b=k} H^{(a,b)}_{\overline{\partial}}$
- Kodaira vanishing for positive line bundles
- ...

In particular, all cohomology groups have at least classical dimension!

Vector bundles and connections

By a *vector bundle* over a *-algebra B, we mean a finitely-generated projective (left) B-module $\mathscr{E}$.

If $\mathscr{E}$ is moreover a B-bimodule and there exists a B-module $\mathscr{E}^\vee$ such that $\mathscr{E} \otimes_B \mathscr{E}^\vee \cong \mathscr{E}^\vee \otimes_B \mathscr{E} \cong B$, then we say that $\mathscr{E}$ is a *line bundle* over B.

Let $(\Omega^\bullet, \mathrm{d})$ be a differential calculus over B and let $\mathscr{F}$ be a left B-module. A *connection on* $\mathscr{F}$ is a $\mathbb{C}$-linear map $\nabla : \mathscr{F} \to \Omega^1 \otimes_B \mathscr{F}$ satisfying

$$\nabla(bg) = \mathrm{d}b \otimes f + b\nabla f, \qquad \text{for every } b \in B, f \in \mathscr{F}.$$

Given a complex structure $\Omega^{(\bullet,\bullet)}$, a *$(0,1)$-connection on* $\mathscr{F}$ is a connection with respect to the differential calculus $(\Omega^{(0,\bullet)}, \overline{\partial})$.

Curvature and holomorphic vector bundles

Any connection extends to a map $\nabla : \Omega^\bullet \otimes_B \mathscr{F} \to \Omega^\bullet \otimes_B \mathscr{F}$ uniquely determined by

$$\nabla(\omega \otimes f) = \mathrm{d}\omega \otimes f + (-1)^{|\omega|}\omega \wedge \nabla f,$$

for every $f \in \mathscr{F}$, and $\omega \in \Omega^\bullet$ a homogeneous form with degree $|\omega|$.

The *curvature* of a connection is the the left B-module map $\nabla^2 : \mathscr{F} \to \Omega^2 \otimes_B \mathscr{F}$. The connection is *flat* if $\nabla^2 = 0$.

A *holomorphic vector bundle over B* is a pair $(\mathscr{F}, \overline{\partial}_{\mathscr{F}})$, where $\mathscr{F}$ is a finitely generated projective left B-module and $\overline{\partial}_{\mathscr{F}} : \mathscr{F} \to \Omega^{(0,1)} \otimes_B \mathscr{F}$ is a flat $(0,1)$-connection on $\mathscr{F}$.

We call $\overline{\partial}_{\mathscr{F}}$ the *holomorphic structure* for $(\mathscr{F}, \overline{\partial}_{\mathscr{F}})$.

Since $\overline{\partial}_{\mathscr{F}} : \mathscr{F} \to \Omega^{(0,1)} \otimes_B \mathscr{F}$ is a flat $(0,1)$-connection, the pair $(\Omega^{(a,\bullet)} \otimes_B \mathscr{F}, \overline{\partial}_{\mathscr{F}})$ is a complex for any fixed $a \in \mathbb{N}$. For $b \in \mathbb{N}$, we denote by $H^{(a,b)}_{\overline{\partial}}(\mathscr{F})$ the b^{th} cohomology group of this complex.

Hermitian vector bundles

An *Hermitian vector bundle* over a *-algebra B is a pair $(\mathscr{F}, h_{\mathscr{F}})$ consisting of a finitely generated projective left B-module $\mathscr{F}$ together with a non-degenerate sesquilinear pairing $h_{\mathscr{F}} : \mathscr{F} \times \mathscr{F} \to B$ satisfying

- $h_{\mathscr{F}}(bf, g) = bh_{\mathscr{F}}(f, g)$ for every $f, g \in \mathscr{F}, b \in B$,
- $h_{\mathscr{F}}(f, g) = bh_{\mathscr{F}}(g, f)^*$ for every $f, g \in \mathscr{F}$,
- for every non-zero $f \in \mathscr{F}$ there exists a non-zero $b \in B$ such that $h_{\mathscr{F}}(f, f) = b^*b$.

Holomorphic Hermitian vector bundles

Note that if $(\Omega^{(\bullet,\bullet)}, \sigma)$ is an Kähler structure, then $(\Omega^\bullet, g_\sigma)$ is an Hermitian vector bundle. In that case, if $(\mathscr{F}, h_{\mathscr{F}})$ is an Hermitian vector bundle over B we can define a sesquilinear map

$$h_{\Omega^\bullet \otimes_B \mathscr{F}} : \Omega^\bullet \otimes_B \mathscr{F} \times \Omega^\bullet \otimes_B \mathscr{F} \to B$$

by putting $h_{\Omega^\bullet \otimes_B \mathscr{F}}(\omega \otimes f, \nu \otimes g) = g_\sigma(\omega h_{\mathscr{F}}(f, g), \nu)$ for every $f, g \in \mathscr{F}, \omega, \nu \in \Omega^\bullet$.

Let $(\mathscr{F}, h_{\mathscr{F}})$ be an Hermitian vector bundle and define

$$\mathfrak{h}_{\mathscr{F}} : \Omega^\bullet \otimes_B \mathscr{F} \to \Omega^\bullet \otimes_B \mathscr{F}, \quad (\omega \otimes f, \nu \otimes g) \mapsto \omega h_{\mathscr{F}}(g, f) \wedge \nu^* .$$

A connection $\nabla : \mathscr{F} \to \Omega^1 \otimes_B \mathscr{F}$ is *Hermitian* if

$$\mathrm{d}\mathfrak{h}_{\mathscr{F}}(f, g) = \mathfrak{h}_{\mathscr{F}}(\nabla(f), 1 \otimes g) + \mathfrak{h}_{\mathscr{F}}(1 \otimes f, \nabla(g)) \text{ for every } f, g \in \mathscr{F} .$$

A *holomorphic Hermitian vector bundle* is a triple $(\mathscr{F}, h_{\mathscr{F}}, \overline{\partial}_{\mathscr{F}})$ such that $(\mathscr{F}, h_{\mathscr{F}})$ and $(\mathscr{F}, \overline{\partial}_{\mathscr{F}})$ is a holomorphic vector bundle

Chern connection and positivity

Edwin Beggs and Shahn Majid

J. Noncommut. Geom. 11 (2017), 669–701

For any Hermitian holomorphic vector bundle $(\mathscr{F}, h_{\mathscr{F}}, \overline{\partial}_{\mathscr{F}})$ there exists a unique Hermitian connection $\nabla : \mathscr{F} \to \Omega^1 \otimes_B \mathscr{F}$ satisfying $\overline{\partial} = (\mathrm{proj}_{\Omega^{(1,0)}} \otimes \mathrm{id}) \circ \nabla$.

We call ∇ the *Chern connection* of $(\mathscr{F}, h_{\mathscr{F}}, \overline{\partial}_{\mathscr{F}})$.

The following was first defined in **A Kodaira Vanishing Theorem for Noncommutative Kahler Structures** Réamonn Ó Buachalla, Jan Stovicek, Adam-Christiaan van Roosmalen arXiv:1801.08125

Definition Let $\Omega^\bullet$ be a differential calculus over a $*$-algebra B, and let $(\Omega^{(\bullet,\bullet)}, \kappa)$ be a Kähler structure for $\Omega^\bullet$. An Hermitian holomorphic vector bundle $(\mathcal{F}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$ is said to be *positive* written $\mathcal{F} > 0$, if there exists $\theta \in \mathbb{R}_{>0}$, such that the Chern connection ∇ of $\mathcal{F}$ satisfies

$$\nabla^2(f) = -\theta \mathbf{i} \kappa \otimes f, \qquad \text{for all } f \in \mathcal{F}.$$

Analogously, $(\mathcal{F}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$ is said to be *negative* written $\mathcal{F} < 0$, if there exists $\theta \in \mathbb{R}_{>0}$, such that the Chern connection ∇ of $\mathcal{F}$ satisfies

$$\nabla^2(f) = \theta \mathbf{i} \kappa \otimes f, \qquad \text{for all } f \in \mathcal{F}.$$

Circle bundle over $\mathcal{O}_q(G/L_s)$

Define $U_q(\mathfrak{l}^s_S) := \langle K_j, E_j, F_j \mid j \in S \rangle \subset U_q(\mathfrak{g})$. Using a similar construction to that of $\mathcal{O}_q(G/L_S)$, we define a quantum homogeneous space

$$\mathcal{O}_q(G/L^s_S) := {}^{U_q(\mathfrak{l}^s_S)}\mathcal{O}_q(G)\,.$$

$\mathcal{O}_q(G/L^s_S)$ admits a free $U(1)$-action whose fixed point subalgebra is given by $\mathcal{O}(G/L_S)$. Thus we think of $\mathcal{O}_q(G/L^s_S)$ as the total space of a principal circle bundle over $\mathcal{O}_q(G/L_S)$.

The induced strong $\mathbb{Z}$-grading $\mathcal{O}_q(G/L^s_S) = \bigoplus_{k\in\mathbb{Z}} \mathcal{E}_k$ then gives a decomposition into a direct sum of $\mathcal{O}_q(G)$-covariant line bundles over $\mathcal{O}_q(G/L_S)$.

It can be shown that all $\mathcal{O}_q(G)$ -covariant line bundles over $\mathcal{O}_q(G/L_S)$ arise in this way.

A well-studied special case is that of $\mathcal{O}_q(\mathbb{CP}^n)$. Here $\mathcal{O}_q(G/L^s_S)$ is the odd-dimensional quantum sphere $\mathcal{O}_q(S^{2n-1})$. Setting $n = 1$, we get the celebrated quantum Hopf fibration over the Podleś sphere.

THEOREM [DG–K–ÓB–S–S]: For every covariant line bundle $\mathcal{E}_k$ over $\mathcal{O}_q(G/L_S)$ there exists a unique covariant $(0,1)$-connection

$$\overline{\partial}_{\mathcal{E}_k} : \mathcal{E}_k \to \Omega^{(0,1)}_q(G/L_S) \otimes_{\mathcal{O}_q(G/L_S)} \mathcal{E}_k \, .$$

Moreover, $\overline{\partial}_{\mathcal{E}_k}$ is flat and hence is an holomorphic structure.

Since each line bundle $\mathcal{E}_k$ has a unique covariant Hermitian structure and hence a Chern connection, we can consider the question of positivity.

q-Deformed invariants

An important point to note is that the numerical invariants given by curvature are *non-classical*. They are q-deformed!

GD–K–ÓB–S–S:

Proposition 5.3. *For any positive line bundle $\mathcal{E}_k$ over quantum projective space $\mathcal{O}_q(\mathbb{CP}^n)$, it holds that*

$$\nabla^2(e) = -(k)_{q^{-2/(n+1)}} \mathbf{i}\kappa \otimes e, \qquad \textit{for all } e \in \mathcal{E}_k,$$

where we have chosen the unique Kähler form κ satisfying

$$\nabla^2(e) = -\mathbf{i}\kappa \otimes e, \qquad \textit{for all } e \in \mathcal{E}_1. \tag{19}$$

A Kodaira Vanishing Theorem for Noncommutative Kahler Structures

Réamonn Ó Buachalla, Jan Stovicek, Adam-Christiaan van Roosmalen

arXiv:1801.08125

Theorem 8.3 (Kodaira Vanishing). *Let $(\Omega^{(\bullet,\bullet)}, \overline{\partial}, \partial)$ be an A-covariant $*$-differential calculus with a complex structure.*

For any positive vector bundle $(\mathcal{F}, \overline{\partial}_{\mathcal{F}}, h)$

$$H^{(a,b)}(\mathcal{F}) = 0, \qquad \text{whenever } a+b>n.$$

Using this one can show that if $\mathscr{F}$ is a negative line bundle over $\mathscr{O}_q(G/L_S)$ we have $H^{(a,b)}_{\overline{\partial}_{\mathscr{F}}} = 0$ whenever $a+b<n$.

Corollary: Let $\mathscr{F}$ be a covariant line bundle over $\mathscr{O}_q(G/L_S)$. Then

- If $H^0_{\overline{\partial}}(\mathscr{E}) \neq 0$ and $H^0_{\partial}(\mathscr{F}) = 0$, then $\mathscr{F} > 0$.
- If $H^0_{\partial}(\mathscr{E}) \neq 0$ and $H^0_{\overline{\partial}}(\mathscr{F}) = 0$, then $\mathscr{F} < 0$.

Results of many hands (Beggs–Majid, Khalkhali–Landi–van Suijlekom, Landi–Moatadelro, Carotenuto–Mrozinski–Ó Buachalla and Díaz García–Ó Buachalla) calculate the degree zero cohomology:

[Borel–Weil] For every irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$ there are $U_q(\mathfrak{g})$ -module isomorphisms

- $H^0_{\bar{\partial}}(\mathcal{E}_k) \cong V_{k\varpi_s}$, for $k \geq 0$, where ϖ_S a fundamental weight. In particular, $H^0_{\bar{\partial}}(\mathcal{E}_k) \neq 0$.
- $H^0_{\bar{\partial}}(\mathcal{E}_k) = 0$, for $k < 0$.

THEOREM [DG–K–ÓB–S–S] For any irreducible quantum flag manifold it holds that, for any $k \in \mathbb{N}$,

- $\mathcal{E}_k > 0$,
- $\mathcal{E}_{-k} < 0$.

Analytic Applications

For any covariant Kähler structure over a compact quantum homogeneous space, the associated Dolbeault–Dirac operator $\overline{\partial} + \overline{\partial}^*$ is essentially self-adjoint, has bounded commutators, and is diagonalisable (Das–Ó Buachalla–Somberg). The compact resolvent condition is more challenging . . .

However, twisting $\overline{\partial} + \overline{\partial}^*$ by a negative line bundle $\mathscr{E} < 0$, and applying the noncommutative version of the Akizuki–Nakano identity

Corollary 6.12 (Akizuki–Nakano identity). *It holds that*

$$\Delta_{\overline{\partial}_{\mathcal{F}}} = \Delta_{\partial_{\mathcal{F}}} + [\mathbf{i}\nabla^2, \Lambda_{\mathcal{F}}].$$

Dolbeault–Dirac Fredholm Operators for Quantum Homogeneous Spaces
Biswarup Das, Réamonn Ó Buachalla, Petr Somberg arXiv:1910.14007

shows that the spectrum has a strictly positive lower bound. From this one can conclude that these operators are Fredholm, making $(\mathscr{O}_q(G/L_S), L^2(\Omega^{(0,\bullet)} \otimes \mathscr{E}_k), \partial_{\mathscr{E}_{-k}} + \overline{\partial}^*_{\mathscr{E}_{-k}})$ excellent candidates for spectral triples.

Thanks for listening!

And happy birthday to Maryam Mirzakhani!