Noncommutative Convexity

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Unital commutative C*-algebras with unital*-homomorphisms		Compact Hausdorff spaces with continuous maps
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Function systems with unital order homomorphisms		Compact convex sets with continuous affine maps
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(Arveson 2007, DK 2015): Proof of existence of Choquet boundary of an operator system.

Subsequently: Major developments in operator spaces/systems. Interesting examples of matrix convex sets arising from e.g. noncommutative real algebraic geometry (Helton-McCullough et al.). But also serious issues e.g. (provably) no good notion of extreme point for matrix convex sets.

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Today: Discuss refined notion of nc convex set, existence of extreme points, categorical equivalence between nc convex sets and operator systems (both unital and non-unital case), nc function theory, noncommutative Choquet theory and application to noncommutative dynamics.

Noncommutative convex sets

Noncommutative convex sets

Let S be a unital operator system (i.e. $1 \in S = S^* \subseteq A$).

Definition

The **nc state space** of S is $K = \coprod_{n \leq \kappa} K_n$,

$$K_n = \{x : A \to M_n \text{ unital completely positive}\},$$

for a suitably large infinite cardinal κ (if S is separable then can take $\kappa = \aleph_0$).

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Refines notion of matrix convex set where $n < \infty$. Subtle but extremely important difference.

Note: each K_n is compact in the point-weak* topology and K is closed under **nc convex combinations**:

$$\sum \alpha_i^* x_i \alpha_i \in K_n$$

for $x_i \in K_{n_i}$ and $\alpha_i \in M_{n,n_i}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$.

Definition (DK2019)

A **compact nc convex set** over a dual operator space E is a graded set $K = \coprod_{n \leq \kappa} K_n$ with $K_n \subseteq M_n(E)$ such that each K_n is compact in the dual topology on $M_n(E)$ and K is closed under nc convex combinations:

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Example

The **nc d-ball** $K = \coprod_{n \leq \aleph_0} K_n$, consisting of "row contractions,"

$$K_n = \{\alpha = (\alpha_1, \dots, \alpha_d) \in M_n^d : \|(\alpha_1, \dots, \alpha_d)\| \leq 1\}.$$

Let $\mathcal{O}_d = \mathrm{C}^*(v_1, \dots, v_d)$ denote the Cuntz algebra. Then K is the nc state space of the Cuntz operator system

$$\mathrm{span}\{1, v_1, v_1^* \dots, v_n, v_n^*\}.$$

Noncommutative functions

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For a unital operator system S with nc state space $K = \coprod K_n$, an element $a \in S$ gives rise to a function $\hat{a} : K \to \coprod M_n$,

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The function $\hat{a}$ is graded, respects direct sums and is equivariant with respect to isometries:

1. $\hat{a}(K_n) \subseteq M_n$ for all n
2. $\hat{a}(\oplus x_i) = \oplus \hat{a}(x_i)$ for all $x_i \in K_{n_i}$
3. $\hat{a}(\alpha^* x \alpha) = \alpha^* \hat{a}(x)\alpha$ for all $x \in K_n$ and isometries $\alpha \in M_{n,m}$

Definition (DK2019)

Let K be a compact nc convex set. A function $f : K \to \coprod M_n$ is an **nc function** if it is graded, respects direct sums and is equivariant with respect to unitaries:

1. $f(K_n) \subseteq M_n$ for all n
2. $f(\oplus x_i) = \oplus f(x_i)$ for all $x_i \in K_{n_i}$
3. $f(\alpha^* x \alpha) = \alpha^* f(x) \alpha$ for all $x \in K_n$ and unitaries $\alpha \in M_n$

The function f is **affine** if in addition it is equivariant with respect to isometries:

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Analogous to notion of nc holomorphic function on nc domain defined by Taylor (1973) and Voiculescu (2000).

We write $\mathrm{C}(K)$ for the C*-algebra of continuous nc functions on K, $\mathrm{A}(K)$ for the unital operator system of continuous affine nc functions on K. Elements in $\mathrm{C}(K)$ are "uniform" limits of nc *-polynomials in $\mathrm{A}(K)$.

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For $a_1, a_2, a_3 \in A(K)$, define $f \in C(K)$ to be the nc polynomial

$$f = a_1 a_2^2 a_3^* - a_1 a_3^* a_2^2.$$

Then for $x \in K_n$,

$$\begin{aligned} f(x) &= a_1(x) a_2^2(x) a_3(x)^* - a_1(x) a_3(x)^* a_2^2(x) \\ &= x(a_1)x(a_2)x(a_2)x(a_3)^* - x(a_1)x(a_3)^* x(a_2)x(a_2) \in M_n. \end{aligned}$$

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Theorem (DK 2019)

We have

$$C(K) = C^*(A(K)) \cong C^*_{max}(A(K)),$$

where $C^*_{max}(A(K))$ is the maximal C-cover of $A(K)$. Moreover, $C(K)^{**}$ is the C*-algebra of bounded nc functions on K.*

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Proof uses noncommutative Gelfand representation theorem of Takesaki (1967) and Bichteler (1969).

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Theorem (DK 2019, Webster-Winkler 1999)

A unital operator system with nc state space K is unitally completely order isomorphic to the operator system $\mathrm{A}(K)$. The category of unital operator systems with unital completely positive maps is (dually) equivalent to the category of compact nc convex sets with continuous affine nc maps:

$$\mathrm{A}(K) \quad \longleftrightarrow \quad K$$

More generally, can consider generalized (i.e. potentially non-unital) operator systems of Werner (2002) and Connes-van Suijlekom (2020).

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A (generalized) **operator system** is a closed self-adjoint subspace of a C*-algebra (i.e. $S = S^* \subseteq A$).

Definition

The **nc quasistate space** of S is the pair (K, z), where $K = \coprod_{n \leq \kappa} K_n$,

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A pair (K, z) consisting of a compact nc convex set K and a point $z \in K_1$ is **pointed** if (K, z) is the nc quasistate space of the operator system $\mathrm{A}(K, z) \subseteq \mathrm{A}(K)$ of functions that vanish at z.

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Theorem (KKM 2021)

An operator system with nc quasistate space (K, z) is isomorphic (completely isometric and completely order isomorphic) to the operator system $\mathrm{A}(K, z)$. The category of operator systems with completely positive maps is (dually) equivalent to the category of pointed compact nc convex sets with continuous pointed affine nc maps:

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Example

Let K be the **nc d-ball**, i.e. the nc state space of the Cuntz operator system

$$\mathrm{span}\{1, v_1, v_1^* \ldots, v_n, v_n^*\}.$$

The extreme points ∂K are irreducible "cuntz isometries" corresponding to irreducible representations of $\mathcal{O}_d$. Note: $\partial K \subseteq K_{\aleph_0}$.

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The extreme points ∂K are irreducible "cuntz isometries" corresponding to irreducible representations of $\mathcal{O}_d$. Note: $\partial K \subseteq K_{\aleph_0}$.

More generally, if S is a unital operator system with nc state space K and $A = \mathrm{C}^*(S)$, then ∂K can be identified with an (often very complicated) subset of the irreducible representations of A.

Review of some classical Choquet theory

Classical Choquet theory: the study of compact convex sets C via the interplay between $A(C)$ and $\mathrm{C}(C)$.

Let C be a compact convex set. A probability measure $\mu \in \mathrm{P}(C) = S(\mathrm{C}(C))$ **represents** a point $x \in C$ if $\mu|_{A(C)} = \delta_x$. Hence

$$x = \int_C y \, d\mu(y).$$

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Theorem (Choquet 1956, Bishop-de Leeuw 1959)

Let C be a compact convex set. For $x \in C$ there is a probability measure μ on C that represents x and is maximal in the **Choquet order***:*

$$\mu \prec \nu \iff \mu(f) \leq \nu(f) \text{ for all convex } f \in C(C)_{sa}.$$

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Maximality in Choquet order is an order-theoretic condition equivalent to $\mathrm{supp}(\mu) \subseteq \partial C$ when C is metrizable (and in an appropriate sense more generally).

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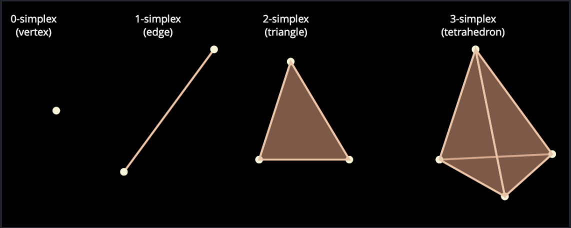
For $C \subseteq \mathbb{R}^d$, Caratheodory's theorem implies there are finitely supported Choquet maximal representing measures for $x \in C$,

$$\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i} \iff x = \sum_{i=1}^{n} \alpha_i x_i,$$

and the above statement is literally true.



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A function system $A(C)$ is a C-algebra if and only if C is a Bauer simplex.*

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More generally, C is a Bauer simplex if and only if it is affinely homeomorphic to the space of probability measures $P(X)$ on a compact Hausdorff space X, i.e. K is the state space of $C(X)$. So Bauer's theorem implies Gelfand's representation theorem.

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It is known that $P(X)^G$ is a simplex. Hence the Choquet-Bishop-de Leeuw integral representation theorem immediately implies the following result:

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Theorem (Ergodic decomposition theorem)

An invariant probability measure can be uniquely decomposed in terms of ergodic probability measures.

Application 3: Dynamical characterization of property (T)

Theorem (Glasner-Weiss 1997)

A group G has property (T) if and only if for every flow (X, G), the set $P(X)^G$ of invariant probability measures is a Bauer simplex.

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By Bauer's theorem, equivalent to the statement that G has property (T) if and only if for every commutative C*-dynamical system $(C(X), G)$, the set $P(X)^G$ of invariant states is the state space of (some) commutative C*-algebra.

Noncommutative Choquet theory

Noncommutative Choquet theory: the study of K or $\mathrm{A}(K)$ via the interplay between $A(K)$ and $C(K)$.

Definition

A self-adjoint nc function $f \in C(K)$ is **convex** if its epigraph

$$\mathrm{Epi}(f) = \coprod_n \{(x,\alpha) : f(x) \leq \alpha\} \subseteq \coprod_n K_n \times M_n$$

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Example

Let $I \subseteq \mathbb{R}$ be a compact interval. Define $K = \coprod K_n$ by

$$K_n = \{\alpha \in (M_n)_{sa} : \sigma(\alpha) \subseteq I\}.$$

Then K is a compact nc convex set with $K_1 = I$. A self-adjoint function $f \in C(K)$ is convex as an nc function iff the restriction $f|_{K_1}$ is operator convex, i.e.

$$f(t\alpha + (1-t)\beta) \leq tf(\alpha) + (1-t)f(\beta)$$

for $t \in [0, 1]$ and self-adjoint $\alpha, \beta \in M_n$ with $\sigma(\alpha), \sigma(\beta) \subseteq I$.

Definition

A self-adjoint nc function $f \in C(K)$ is **convex** if its epigraph

$$\mathrm{Epi}(f) = \coprod_n \{(x,\alpha) : f(x) \leq \alpha\} \subseteq \coprod_n K_n \times M_n$$

is an nc convex set.

Equivalently, $f(\alpha^* x \alpha) \leq \alpha^* f(x) \alpha$ for all $x \in K$ and all isometries α.

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Essentially the Hansen-Pedersen-Jensen inequality.

An nc state $\mu : C(K) \to M_n$ **represents** a point $x \in K$ if $\mu|_{A(K)} = \delta_x$, i.e.

$$\mu(a) = a(x), \quad \text{for all } a \in \mathrm{A(K)}.$$

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Theorem (NC Choquet-Bishop-de Leeuw - DK 2019)

For $x \in K$ there is an nc state $\mu : C(K) \to M_n$ that represents x and is maximal in the **nc Choquet order***:*

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Theorem (DK 2019)

Maximality in the nc Choquet order implies that $\operatorname{supp}(\mu) \subseteq \partial K$ *in an appropriate sense.*

In the separable setting we obtain an integral representation theorem.

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Theorem (Noncommutative integral representation - DK 2019)

For $x \in K$ there is a nc probability measure λ on K that represents x and is supported on ∂K, meaning that

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Idea: An nc measure is a cp map valued measure. For $f \in C(K)$ and an nc measure λ on K,

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More generally, obtain integral representations of nc states on $C(K)$. Applies to e.g. ucp maps on C*-algebras.

Noncommutative Choquet simplices

Definition

A compact nc convex set K is an **nc (Choquet) simplex** if each point $x \in K$ has a unique nc state $\mu_x : \mathrm{C}(K) \to M_n$ that represents it and is maximal in the nc Choquet order.

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Hence generalizes classical simplices.

Theorem (KS 2019)

The following are equivalent for a compact nc convex set K:

1. K is an nc Choquet simplex
2. $A(K)^{**}$ is a von Neumann algebra, i.e. $A(K)$ is a **C*-system** in the terminology of Kirchberg-Wassermann
3. $A(K)$ is (c,max)-nuclear in the sense of Kavruk-Paulsen-Todorov-Tomforde, i.e.

$$A(K) \otimes_c S = A(K) \otimes_{max} S$$

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Generalization of the fact that a compact convex set C is a simplex if and only if $\mathrm{A}(C)^{**}$ is a von Neumann algebra.

Corollary (KS 2019)

Let S be an operator system with nc state space K. If S is a C*-algebra or has the weak expectation property (in particular if it is nuclear) then K is an nc simplex.

Application 1: State spaces of C*-algebras

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A nc Choquet simplex K is an **nc Bauer simplex** if ∂K is closed.

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Theorem (KKM 2021, KS 2019)

1. *A compact nc convex set K is affinely homeomorphic to the nc state space of a unital C*-algebra if and only if it is an nc Bauer simplex.*
2. *A pointed compact nc convex set (K, z) is affinely homeomorphic ot the nc quasistate space of a C*-algebra if and only if K is an nc Bauer simplex and $z \in \partial K$.*

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Compare to (deep) characterization of state spaces of C*-algebras by Alfsen-Shultz (1978) in terms of compact convex sets with an orientation.

Application 2: Noncommutative ergodic decomposition theorem

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The set K^G of invariant nc states (resp. nc quasistates) is an nc Choquet simplex.

An invariant nc state $\mu \in K^G$ is **ergodic** if $\mu \in \partial(K^G)$. The nc Choquet-Bishop-de Leeuw theorem implies an ergodic decomposition theorem in this setting.

Application 3: Noncommutative dynamical characterization of property (T)

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Theorem (KS 2019)

A group G has property (T) if and only if whenever (A, G) is a C-dynamical system with nc state space K (resp. nc quasistate space (K, z)), the set K^G of invariant nc states is an nc Bauer simplex, and hence affinely homeomorphic to the nc state space of (some) C*-algebra.*

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Corollary

A group G has property (T) if and only if whenever (A, G) is a C-dynamical system, then the set K_1^G of invariant states (resp. quasistates) is the state space (resp. quasistate space) of a C*-algebra.*

Thanks!