Compact Quantum Metric Spaces

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Preface

We are immensely grateful for the opportunity to give a mini-course at the Hausdorff school for advanced studies in mathematics, during the program on *Noncommutative Geometry and Operator Algebras*, organized by the Hausdorff Center for Mathematics from May 2nd to May 5th 2023, at the University of Bonn. Our course intended to introduce the theory of *quantum metric spaces*, with a focus on the notion of convergence for quantum metric structures. We are very hopeful that our lectures helped kindle interest in this relatively new subject, about which much remain to be explored.

We chose to focus on core topics of the subject, in order to provide the basic ideas found in the literature on quantum metric spaces, while also touching on very recent developments, particularly, the spectral propinquity. Our hope is that the chosen topics will both prove helpful getting into the subject, while giving some ideas on its current research direction. Of course, our choices are very much informed by our own perspective on the field, and the necessity of selecting only a few topics in the field in order to create a reasonable mini-course implies that many interesting and deep results in the literature will not be addressed. This includes, for instance, work on quantum group and related structures, or much of our work on the covariant propinquity. Nonetheless, we are hopeful that this course will make the work we could not address here accessible.

The following set of notes was written in support of our minicourse. As the minicourse, these notes do not main at being exhaustive, but instead, to provide a decent overview of the basic ideas of noncommutative metric geometry. One of our goal is also to streamline some of the terminology and reorder certain concepts, as a decade has past with various developments, giving us an evolving perspective. We are very interested in answering any question which these notes may raise, and we are happy to discuss any further inquiry in noncommutative metric geometry. We more than welcome comments and suggestions about these notes, as well as the likely corrections for the typos which inevitably sneaked in these notes in spite of our best efforts.

Once more, we are deeply thankful for this great opportunity, and we hope our minicourse proved interesting.

Introduction

The founding allegory of noncommutative geometry is that certain algebras can be understood as noncommutative analogues of various algebras of functions over spaces endowed with geometric data, as if these noncommutative algebras give us a glimpse of a quantum space which can only be described via their observables, whose lack of commutativity encodes the uncertainty principle: not all can be seen at once. We thus must learn to study the geometry of such quantum objects by extending various constructions from the classical world to algebras of functions, in an often nontrivial way, so that the constructions remain meaningful when we drop commutativty, and opens new examples out of reach from the classical picture. This approach has proven beneficial in the study of singular spaces obtained from geometric settings — from orbit spaces to fractals, from the space of foliations to spaces of tilings — and also to the study of noncommutative algebras, with, for instance, the export from the topological world of such tools as K-theory, K-homology, and other now common tools of C*-algebra theory. Our purpose in these notes is to develop the theory of noncommutative analogues of the algebras of Lipschitz functions ovre compact metric spaces. Our focus is the study of the geometry of the hyperspace of all quantum compact metric spaces, and derive properties of quantum compact metric spaces from their belonging to the closure of certain classes of spaces.

The geometry defined on the hyperspace of all quantum compact metric spaces enables us to formalize certain common heuristics in mathematical physics. Mathematical physics raises many questions about behaviors of various models as certain parameters, be it a deformation parameter or the dimension of a matrix model, are taken to some limit. For instance, a quantized version $\mathfrak{A}_\hbar$ of some commutative algebra $C(M)$, where M is a compact manifold, given as some C*-algebra, may be expected to converge in some sense to $C(M)$ as $\hbar$ is taken to 0. Another such heuristic is given by the convergence, as the dimension n tends to infinity, of the C*-algebra $\mathfrak{A}_n$ generated by two unitaries:

$$S_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \text{ and } C_n = \begin{pmatrix} 1 & & & \\ & \exp\left(\frac{2i\pi}{n}\right) & & \\ & & \ddots & \\ & & & \exp\left(\frac{2i(n-1)\pi}{n}\right) \end{pmatrix}, \quad (0.0.1)$$

where $C_n S_n = \exp(\frac{2i\pi}{n}) S_n C_n$, to the C*-algebra $C(\mathbb{T}^2)$ of continuous functions over the 2-torus, i.e. the universal C*-algebra generated by 2 commuting unitaries. Similar matrix models called fuzzy spheres are expected to converge to the C*-algebra $C(S^2)$ of continuous functions over the 2-sphere. More generally, the construction of new physical models as limits of simpler models, specified over finite space times or lattices inside the continuum limit, is a natural approach.

The study of various notions of limits of C*-algebras is deeply rooted in the field of operator algebra theory. Inductive limits provide a categorical perspective which is particularly well suited to homological considerations — in particular K-theory and its decendent — while continuous fields of C*-algebras have proven an efficient way to describe certain C*-algebras, such as certain C*-crossed-products, and to formalize the idea of quantization. Our approach, motivated by its potential physical relevance, but also by developments in Riemannian geometry, is to try and endow a space of certain noncommutative algebras with an actual metric, up to an appropriate notion of isomorphism, so that the entire machinery of analysis can be brought to bare on problems involving limits and approximation questions with C*-algebras. As we shall see, some important properties are indeed continuous for some of our metrics, such as spectra of spectral triples, thus illustrating some of the potential for this new approach.

Our story thus begins with the search for noncommutative analogue of a compact metric space. Indeed, almost since their inception, metric spaces have provided a natural framework to study how far two subspaces may be, via the Hausdorff distance [21]. The intrinsic version of the Hausdorff distance, due to Edwards [15], was introduced in order to topologize the space of all "space-times" as a step toward formalizing an approach to quantum gravitation suggested by Wheeler [59]. This intrinsic version, now known as the Gromov-Hausdorff distance, was extended to the class of proper metric spaces by Gromov [19, 20] in the context of geometric group theory, and has found many applications in Riemannian geometry since. Thus, a template exists for defining a metric on the space of noncommutative metric spaces, if we indeed find an adequate notion of quantum compact metric space.

Connes' original idea of a quantum compact metric space was part of the motivation stated in [11] for the introduction of spectral triples. Spectral triples have emerged as the preferred means to define a noncommutative Riemannian manifold, and thus they will occupy a central place in our work. They are given as triples $(\mathfrak{A},\mathscr{H},\not{D})$ of a C*-algebra $\mathfrak{A}$, a Hilbert space $\mathscr{H}$ on which $\mathfrak{A}$ acts, and a self-adjoint operator $\not{D}$ on some dense subspace of $\mathscr{H}$ which boundedly commutes with a dense *-subalgebra of $\mathfrak{A}$ and has a compact resolvent. While spectral triples are probably best understood as unbounded analogues of Fredholm modules, i.e. as abstraction of differential elliptic operators for which a form of index theory may be devised, they also notably allow for the definition of an extended pseudo-metric on the state space of their underlying algebras, known as the Connes' metric. If $(\mathfrak{A},\mathscr{H},\not{D})$ is a spectral triple, Connes' metric between any two states φ,ψ of $\mathfrak{A}$ is defined by:

$$\mathsf{mk}_{\not{D}}(\varphi,\psi) := \sup\left\{|\varphi(a)-\psi(a)| : a\,\mathrm{dom}\,(\not{D}) \subseteq \mathrm{dom}\,(\not{D}), |||[\not{D},a]|||_{\mathscr{H}} \leqslant 1\right\},$$

where $|||\cdot|||_{\mathscr{H}}$ is the operator norm. Thus, a general view of what quantum compact metric spaces started to emerge. The question of what topology Connes' metric induces on the state space remained largely unaddressed at first. Whatever it may be, it is always stronger than the weak* topology, as convergence for the Connes metric implies by definition uniform convergence on some total subset of the underlying C*-algebra. It also always induces a Hausdorff topology. As a result, since the state space of a unital C*-algebra is weak* compact, the smallest topology that the Connes metric may induce, is also the only possible compact topology this metric may induce, and it is the weak* topology.

There are several natural reasons to want to work with spectral triples whose Connes' metric induces the weak* topology. As we just noted, it is the only way to make the state space compact for this metric. In turn, this means that we can now work with compact metric spaces, which is very well suited for the development of a Hausdorff-type metric. Moreover, it is the natural topology to employ for applications, as it is the natural topology in probability theory for convergence in distribution, and it is the topology one gets if one defines neighborhood of states by asking states to be close when their measurements on finitely many observables is within some open sets of scalars. Furthermore, it is a canonical topology to choose from a functional analytic point of view, both because of its compactness — the very reason to work with the weak* topology is to regain compactness in infinite dimension — and because it is indeed the topology one would obtain in the classical picture. More specifically, if M is a connected compact Riemannian spin manifold, the prototype of a spectral triple is given by $(C(M), \Gamma^2(SM), \not{D})$, where SM is the spinor bundle of M, Γ^2 the Hilbert space of square-integrable sections of SM, and $\not{D}$ is the Dirac operator of M. In this case, for any $f \in C^1(M)$, the operator $[\not{D}, f]$ acts as the Clifford multiplication by the gradient of f. Connes' distance between two states φ, ψ of $C(M)$ becomes:

$$\mathsf{mk}_{\not{D}}(\varphi, \psi) := \sup\left\{|\varphi(f) - \psi(f)| : f \in C(M), f \text{ is } 1 - \text{Lipschitz}\right\}.$$

Therefore, Connes' metric becomes a special case of the Monge-Kantorovich metric in this classical picture. In particular, it induces the path metric on M induced by the Riemannian metric of M, and it induces the weak* topology on the space of regular probability measures on M, i.e. on the state space of $C(M)$.

More generally, if (M, d) is a compact metric space, then the metric d induces a natural seminorm defined on a dense subalgebra of $C(M)$, called the *Lipschitz seminorm*, which to every $f \in C(M)$, associated the Lipschitz constant:

$$\mathsf{L}_d(f) := \sup\left\{\frac{|f(x) - f(y)|}{d(x, y} : x, y \in M, x \neq y\right\},$$

allowing for the value ∞. The fact the domain of this seminorm, i.e. the subspace of $C(M)$ where it is finite, is dense, is an immediate consequence of the Stone-Weierstraßtheorem, owing to the facts that this domain is an algebra — thanks to the Leibniz inequality satisfied by L_d — and it contains the functions $x \in M \mapsto d(x, y)$ for all $y \in M$, which separate the points of M, as well as the constants. Moreover, the set of Lipschitz functions of Lipschitz constant at most 1 is closed under pointwise — let alone, uniform — convergence in $C(M)$. A deep result is that, if we fix $x \in M$, since the set $\{f \in C(M) : \mathsf{L}_d(f) \leqslant 1, f(x) \leqslant 0\}$ is an equicontinuous family of functions over the compact space (M,d), with valued in the closed ball in $\mathbb{C}$ of center 0 and radius the diameter of (M,d), then the Arzéla-Ascoli theorem implies that this set is totally bounded in $C(M)$. As it is closed as well, and as $C(M)$ is complete, we conclude that $\{f \in C(M) : \mathsf{L}_d(f) \leqslant 1, f(x) \leqslant 0\}$ is compact, for any choice of $x \in M$. In turn, we can use the compactness of $\{f \in C(M) : \mathsf{L}_d(f) \leqslant 1, f(x) \leqslant 0\}$ to prove that the Monge-Kantorovich metric defined between any two states $\varphi, \psi \in C(M)$ by:

$$\mathsf{mk}_{\mathsf{L}}(\varphi,\psi) := \sup\left\{|\varphi(f) - \psi(f)| : f \in C(M), \mathsf{L}_d(f) \leqslant 1\right\}$$

induces the weak* topology on the state space of $C(M)$. As we shall see later in these notes, this property is in fact equivalent to the compactness of $\{f \in C(M) : \mathsf{L}_d(f) \leqslant 1, f(x) \leqslant 0\}$. Moreover, since $x \in M \mapsto d(x,y)$ is 1-Lipschitz for any $y \in M$, we see that, after identifying points of M with the characters of $C(M)$ given by evaluation maps, $\mathsf{mk}_{\mathsf{L}_d}(x,y) = d(x,y)$. Thus, the restriction of $\mathsf{mk}_{\mathsf{L}_d}$ to the space of characters of $C(M)$, i.e. to the Gelfand spectrum of $C(M)$, topologized by the weak* topology, which is of course homeomorphic to M, is a metric isometric to d. So the Lipschitz seminorm encodes the metric on M, and it has several notable topological properties which would make sense even over a noncommutative algebra. We note in passing that we could as well as worked with the self-adjoint subalgebra of $C(M)$, i.e. with real valued continuous functions, for this discussion.

Adopting this perspective, Rieffel [52, 53, 54] proposed to define a quantum compact metric space as a pair $(\mathfrak{A}, \mathsf{L})$, where L is a seminorm which shares the above listed properties of a Lipschitz seminorms, and $\mathfrak{A}$ is a space of the same "type" as the space of real-valued continuous functions over compact metric spaces — in practice, Rieffel worked with $\mathfrak{A}$ being order unit spaces (not even assumed to be complete), that is, up to isometric positive linear isomorphisms, with subspaces of the space of self-adjoint elements of a unital C*-algebra. In time, the development of analogues of the Gromov-Hausdorff distance to quantum compact metric spaces has revealed that it is helpful to strengthen the initial definition of Rieffel. For example, additional Leibniz-type inequalities were required by Rieffel in [55, 56] when working toward a convergence of modules. In particular, this required one to work with C*-algebras, but created difficulties with the triangle inequality for the quantum Gromov-Hausdorff distance. Other reasons to modify the quantum Gromov-Hausdorff distance — and adjust accordingly the notion of quantum compact metric spaces — include devising a distance for which distance zero implies *-isomorphism of the underlying C*-algebras; such constructions include Kerr's distance [29] where a quantum compact metric space is taken as a pair of an operator system and an analogue of the Lipschitz seminorm, or the nuclear distance of Li [49], for instance. Other constructions from Wu [60, 61] used the notion of matricial Lipschitz seminorms. However, several of these modifications to Rieffel's original construction encountered various difficulties: Kerr's distance, for instance, is complete under some assumptions which are not compatible with it satisfying the triangle inequality. In general, there is a cost in working outside of the category of C*-algebras when defining quantum compact metric spaces, as it makes it more difficult to discuss such notions as modules, actions by an appropriate notion of automorphisms, or spectral triples.

The *propinquity*, is an analogue of the Gromov-Hausdorff propinquity well adapted to working with C*-algebras endowed with Lipschitz-like seminorms which satisfy a form of the Leibniz identity. The propinquity is defined on the class of quantum compact metric spaces given as pairs of a unital C*-algebra and a Lipschitz-like seminorm. It is a complete metric, up to full quantum isometry — in particular, distance zero implies that the underlying C*-algebras are *-isomorphic. Its restriction to the class of classical compact metric spaces induces the same topology as the Gromov-Hausdorff distance. It is indeed possible to prove that fuzzy tori converge to (quantum) tori [34], fuzzy spheres to actual spheres [57], AF algebras form various continuous families [2], various deformations of quantum tori form continuous families [35], noncommutative solenoids are limits of quantum tori [48, 7], and more. We proved an analogue of the Gromov compactness theorem for the propinquity [39], thus exhibiting compact classes, for instance, of AF algebras, and also providing the proof that indeed, the topology induced by the propinquity in the classical setting is the topology of the Gromov-Hausdorff distance.

Moreover, the definition of the propinquity in [36] is flexible enough to allow for its generalization to structures associated to quantum compact metric spaces. In [41, 46], a Gromov-Hausdorff like distance was defined between metrical C*-correspondence to formalize the idea of convergence of quantum vector bundles, with application, for instance, to the Heisenberg modules over quantum tori [44, 42], using the connections studied by Connes [10] and Rieffel [12]. The metrical propinquity thus defined is indeed complete, zero exactly between fully quantum isometric metrical C*-correspondences, and restricts to give the same topology as the propinquity when we look at a C*-algebra $\mathfrak{A}$ as a $\mathbb{C}$-$\mathfrak{A}$-C*-correspondence.

Another modification of the propinquity is the covariant propinquity [43, 40], later extended to metrical C*-correspondences [47]. This metric is now defined between actions of proper monoids on quantum compact metric spaces and metrical C*-correspondences, where the actions are by Lipschitz morphisms, and the monoids are allowed to be different. One motivation for such a construction is the good behavior of group actions under convergence for the propinquity [3]. The covariant propinquity provides a metric framework for the study of quantum dynamics. It also can be used to obtain certain results on the closure of classes of quantum compact metric spaces, and study the properties of isometry groups for quantum compact metric spaces and spectral triples [25].

Among the main examples of quantum compact metric spaces are those given by certain spectral triples: when the Connes' distance of a spectral triple induces the weak* topology on the state space of the underlying C*-algebra, it is called metric. Examples include quantum tori [52], C*-crossed-products [22, 30], Podles spheres [1], fractals [9, 32], AF algebras [5], Bunce-Deddens algebras and noncommutative solenoids [7], and more. The spectral propinquity is in fact, a special case of the covariant propinquity of metrical C*-correspondences, which give a metric, up to unitary equivalence, on the class of metric spectral triples. The spectral propinquity has some strong properties, such as continuity of the spectrum and of the bounded continuous functional calculus [33]. It can be used to formalize the heuristics that certain matrix models converge to models over quantum tori [45], or that the spectral triples on fractals from [9] is indeed a limit of spectral triples on graphs [31].

The functional analytic framework we developed for the Gromov-Hausdorff propinquity thus opens the possibility to apply new methods inspired from metric geometry in noncommutative geometry. These notes present a synthesis of the main results about the propinquity, in the hope that the reader can approach the literature on the subject and carry out research in this new, as of yet largely unexplored, domain.

Chapter One

The Category of Compact Quantum Metric Spaces

1.1 COMPACT QUANTUM METRIC SPACES

A *quantum compact metric space* is a noncommutative generalization of the algebra of Lipschitz functions over a compact metric space.

Definition 1.1.1. A *K-compact quantum metric space* $(\mathfrak{A}, \mathsf{L})$, where $K \in [1,\infty)$, is an ordered pair consisting of a unital C*-algebra $\mathfrak{A}$, and a seminorm L, defined on a Jordan-Lie subalgebra $\mathrm{dom}(\mathsf{L})$ of the space $\mathfrak{sa}(\mathfrak{A})$ of self-adjoint elements of $\mathfrak{A}$, such that

1. the *Monge-Kantorovich metric*, defined on the state space $\mathscr{S}(\mathfrak{A})$ of $\mathfrak{A}$, by setting, for any two states φ,ψ of $\mathfrak{A}$:
$$\mathsf{mk}_{\mathsf{L}}(\varphi,\psi) = \sup\left\{\left|\varphi(a)-\psi(a)\right| : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\right\}, \tag{1.1.1}$$
metrizes the weak* topology on the state space $\mathscr{S}(\mathfrak{A})$ of $\mathfrak{A}$,
2. $\mathsf{L}(1_{\mathfrak{A}}) = 0$,
3. the unit ball $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1\}$ of L is closed for $\|\cdot\|_{\mathfrak{A}}$ in $\mathfrak{sa}(\mathfrak{A})$,
4. for all $a, b \in \mathrm{dom}(\mathsf{L})$, the following inequality holds:
$$\max\left\{\mathsf{L}\left(\frac{ab+ba}{2}\right), \mathsf{L}\left(\frac{ab-ba}{2i}\right)\right\} \leqslant K\left(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{A}} + \mathsf{L}(a)\mathsf{L}(b)\right). \tag{1.1.2}$$

When $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space, the seminorm L is called a *Lipschitz seminorm* on $\mathfrak{A}$, or an *L-seminorm* on $\mathfrak{A}$ for short. Elements of $\mathrm{dom}(\mathfrak{A})$ are then called *Lipschitz elements* of $(\mathfrak{A}, \mathsf{L})$.

Convention 1.1.2. When L is a seminorm defined on some subspace $\mathrm{dom}(\mathsf{L}) \subseteq E$ of a vector space E, we set $\mathsf{L}(x) = \infty$ whenever $x \in E \setminus \mathrm{dom}(\mathsf{L})$ (where $\infty \geqslant x$ for all $x \in \mathbb{R}$). With this convention, the domain $\mathrm{dom}(\mathsf{L})$ of L is the set $\{x \in E : \mathsf{L}(x) < \infty\}$. We will work with this convention from now on; for calculations, we set $0 \cdot \infty = 0$ and $\infty + x = x + \infty = \infty$ for all $x \in [0,\infty]$.

In particular, if $\{x \in E : \mathsf{L}(x) \leqslant 1\}$ is closed, then since L is a seminorm, $\{x \in E : \mathsf{L}(x) \leqslant t\}$ is closed for all $t \in \mathbb{R}$, and thus L, as a $[0,\infty]$-valued function, is lower semi-continuous on E.

Remark 1.1.3. The assumption on the Lipschitz seminorm L of a quantum compact metric space $(\mathfrak{A},\mathsf{L})$ implies that it is lower semicontinuous over $\mathfrak{sa}(\mathfrak{A})$, and therefore, in particular, for all $a \in \mathfrak{sa}(\mathfrak{A})$, if $(a_n)_{n\in\mathbb{N}}$ is a sequence converging to a in $\mathfrak{sa}(\mathfrak{A})$, then:

$$\mathsf{L}(a) \leqslant \liminf_{n\to\infty} \mathsf{L}(a_n).$$

Notation 1.1.4. Given a C*-algebra $\mathfrak{A}$, we write $\Re a := \frac{1}{2}(a + a^*)$ and $\Im a := \frac{1}{2i}(a - a^*)$, so $\Re a, \Im a \in \mathfrak{sa}(\mathfrak{A})$ and $a = \Re a + i\Im a$. For any $a, b \in \mathfrak{sa}(\mathfrak{A})$, the Jordan product and Lie product of a and b are then just $\Re(ab)$ and $\Im(ab)$, respectively.

Notation 1.1.5. $(\mathfrak{A},\mathsf{L})$ is a quantum compact metric space when there exists $K \geqslant 1$ such that $(\mathfrak{A},\mathsf{L})$ is a K-quantum compact metric space.

Definition 1.1.6. A (C,D)-quantum compact metric space $(\mathfrak{A},\mathsf{L})$ is a quantum compact metric space such that, for all $a, b \in \mathrm{dom}(\mathsf{L})$,

$$\max\{\mathsf{L}(\Re(ab)),\mathsf{L}(\Im(ab))\} \leqslant C(\mathsf{L}(a)\,\|b\|_{\mathfrak{A}} + \mathsf{L}(a)\,\|b\|_{\mathfrak{A}}) + D\mathsf{L}(a)\mathsf{L}(b).$$

A Leibniz quantum compact metric space is a $(1,0)$-quantum compact metric space.

The classical model of a quantum compact metric space is given as follows.

Example 1.1.7 (Fundamental Example). Let (X,d) be a compact metric space. For each $f : X \to \mathbb{R}$, we define the *Lipschitz constant* $\mathsf{L}(f)$ of f as

$$\mathsf{L}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)} : x, y \in X, x \neq y\right\},$$

allowing for the value ∞.

The pair $(C(X),\mathsf{L})$ is an example of a quantum compact metric space, see for instance [14]. We will recover this fact when we establish Theorem (1.1.19). The metric mk_{L} was introduced by Kantorovich [27] as part of their study of Monge's transportation problem. The formulation which we use here is due to Kantorovich and Rubinstein [28]. This metric has many different names, and in particular, is sometimes called the Wassertein metric; Wassertein re-introduced this metric within the realm of probability in [58], and was named the Wasserstein distance by Dobrushin in [13].

This metric has many applications, precisely owing to the fact that it metrizes the weak* topology (when working with compact metric spaces). Moreover, we note that if $x, y \in X$, identifying points with characters of $C(X)$, then $\mathsf{mk}_{\mathsf{L}}(x,y) = d(x,y)$ by construction (noting that $t \in X \mapsto d(x,t)$ is 1-Lipschitz). Therefore, *we may recover the distance d on X from the Lipschitz seminorm* L.

Definition (1.1.1) thus is our attempt to capture the properties which make the Lipschitz seminorm L interesting among all densely-defined seminorms on $C(X)$, while allowing for enough generality to find interesting noncommutative examples.

Finite dimensional C*-algebras also provide many examples of quantum compact metric spaces.

Example 1.1.8. Let $\mathfrak{A}$ be a finite dimensional C*-algebra. For all $a \in \mathfrak{sa}(\mathfrak{A})$, set

$$\mathsf{L}(a) = \inf_{t\in\mathbb{R}} \|a - t1\|_{\mathfrak{A}}.$$

The pair $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space. Indeed, mk_{L} simply induces the norm topology on $\mathscr{S}(\mathfrak{A})$, which equals the weak* topology since $\mathfrak{A}$ is finite dimensional.

Our first result about quantum compact metric space shows that the Monge-Kantorovich metric allows us to recover the Lipschitz seminorm, so something of the duality between metric and Lipschitz seminorm remain in the noncommutative realm, albeit not quite in its original form. This theorem is, however, instrumental to understand morphisms between quantum compact metric spaces.

Theorem 1.1.9 ([54])**.** *If* $(\mathfrak{A}, \mathsf{L})$ *is a quantum compact metric space, then for all* $a \in \mathfrak{sa}(\mathfrak{A})$*:*

$$\mathsf{L}(a) = \sup\left\{ \frac{|\varphi(a) - \psi(a)|}{\mathsf{mk}_{\mathsf{L}}(\varphi, \psi)} : \varphi, \psi \in \mathscr{S}(\mathfrak{A}), \varphi \neq \psi \right\}, \tag{1.1.3}$$

allowing for the value ∞*.*

Proof. Let us write $\mathsf{S}(a) := \sup\left\{ \frac{|\varphi(a)-\psi(a)|}{\mathsf{mk}_{\mathsf{L}}(\varphi,\psi)} : \varphi, \psi \in \mathscr{S}(\mathfrak{A}), \varphi \neq \psi \right\}$ for all $a \in \mathfrak{sa}(\mathfrak{A})$, allowing for ∞.

By Definition (1.1.1), we immediately observe that for all $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ with $\varphi \neq \psi$,

$$|\varphi(a) - \psi(a)| \leqslant \mathsf{L}(a)\mathsf{mk}_{\mathsf{L}}(\varphi, \psi)$$

(note that this is trivial when $\mathsf{L}(a) = \infty$), and thus $\mathsf{S}(a) \leqslant \mathsf{L}(a)$.

Let now $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{S}(a) = 1$. Therefore, for all $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$, we have $|\varphi(a) - \psi(a)| \leqslant \mathsf{mk}_{\mathsf{L}}(\varphi, \psi)$.

Let now $E := \{\mu \in \mathfrak{A}' : \mu(1) = 0\}$; E is the dual of ${}^{\mathfrak{sa}(\mathfrak{A})}\!/_{\mathbb{R}1}$. Let $\mathsf{L}' : \mu \in E \mapsto \sup\{|\mu(a)| : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$.

Let $E_2 = \{\mu \in E : \|\mu\|_{\mathfrak{A}'} \leqslant 2\}$. Note that, for any two states $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$, we have $\varphi - \psi \in E_2$. Conversely, if $\mu \in E_2$, then, by the Hahn-Jordan decomposition theorem, $\mu = \theta - \sigma$ for two positive linear functionals θ and σ over $\mathfrak{sa}(\mathfrak{A})$. such that $\|\theta\|_E + \|\sigma\|_E = \|\mu\|_E = 2$. Since $\mu(1) = 0$, we have $\theta(1) = \sigma(1)$. Let $t = \theta(1)$. If $t = 0$, let φ be any state of $\mathfrak{A}$, and set $\psi := \varphi$. Otherwise, set $\varphi := \frac{1}{t}\theta$ and $\psi := \frac{1}{t}\sigma$. Either way, $\mu = t\varphi - t\psi$ with $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ and $t \in [0,1]$. Thus $\mu = \varphi - ((1-t)\varphi + t\psi)$, with $\varphi, (1-t)\varphi + t\psi \in \mathscr{S}(\mathfrak{A})$. All in all, we see that $E_2 = \{\varphi - \psi : \varphi, \psi \in \mathscr{S}(\mathfrak{A})\}$.

Therefore, if $\mu \in E_2$, then $|\mu(a)| = |\varphi(a) - \psi(a)| \leqslant \mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \mathsf{L}'(\mu)$ for some $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$. In summary, if $a \in \mathfrak{sa}(\mathfrak{A})$ and $\mathsf{S}(a) \leqslant 1$, then $|\mu(a)| \leqslant \mathsf{L}'(\mu)$ for all $\mu \in E_2$. Therefore by homogeneity, for all $\lambda \in E$, we have $|\lambda(a)| \leqslant \mathsf{L}'(\lambda)$ for all $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{S}(a) \leqslant 1$. In particular, if $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{S}(a) \leqslant 1$, then for all $\lambda \in E$ with $\mathsf{L}'(\lambda) \leqslant 1$, we have $|\lambda(a)| \leqslant 1$. So a is in the prepolar of $\{\lambda \in E : \mathsf{L}'(\lambda) \leqslant 1\}$. On the other hand, $\{\lambda \in E : \mathsf{L}'(\lambda) \leqslant 1\}$ is by definition the polar of $\{b \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(b) \leqslant 1\}$. So, by the bipolar theorem, we conclude that if $a \in \mathfrak{sa}(\mathfrak{A})$ and $\mathsf{S}(a) \leqslant 1$, then a belongs to the bipolar of $\{b \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(b) \leqslant 1\}$. Since $\{b \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(b) \leqslant 1\}$ is, by assumption, balanced, convex and closed, it is equal to its bipolar. So if $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{S}(a) \leqslant 1$, then $a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1$, as needed. So $\mathsf{L}(a) \leqslant 1$. Therefore, $\mathsf{L} \leqslant \mathsf{S}$. We conclude that $\mathsf{L} = \mathsf{S}$ as needed. □

Remark 1.1.10. We may *not* replace the state space in Expression (1.1.3) of Theorem (1.1.9) with its subset of extreme points in general, although it is indeed true in the classical case of Example (1.1.7). Unfortunately in general, one may not recover mk_{L} from its restriction to the set of extreme states. See, for instance, [53], for a detailed overview.

We now establish some basic properties of quantum compact metric spaces, progressing toward a characterization which will help us establish new examples. We begin with the natural observation that the domain of a Lipschitz seminorm needs to be dense in the space of self-adjoint operators.

Proposition 1.1.11. *If* $(\mathfrak{A}, \mathsf{L})$ *is a quantum compact metric space, then* $\mathrm{dom}(\mathsf{L})$ *is dense in* $\mathfrak{sa}(\mathfrak{A})$.

Proof. Assume that $\mathrm{cl}(\mathrm{dom}(\mathsf{L})) \subsetneq \mathfrak{sa}(\mathfrak{A})$. By Hahn-Banach theorem, applied to the real vector space $\mathfrak{sa}(\mathfrak{A})$, there exists a continuous linear functional $\mu \in \mathfrak{sa}(\mathfrak{A})'$ on $\mathfrak{sa}(\mathfrak{A})$ such that $\mu \neq 0$, yet $\mu(\mathrm{cl}(\mathrm{dom}(\mathsf{L}))) = 0$. Up to re-scaling μ by $\frac{2}{|||\mu|||}$, we assume that $|||\mu||| = 2$. By Hahn-Jordan decomposition, there exists two positive continuous linear functionals $\varphi, \psi \in \mathfrak{sa}(\mathfrak{A})'$ on $\mathfrak{sa}(\mathfrak{A})$ such that $\mu = \varphi - \psi$, with $|||\varphi||| + |||\psi||| = |||\mu||| = 2$. Now, since $1 \in \mathrm{dom}(\mathsf{L})$, we conclude that $0 = \mu(1) = \varphi(1) - \psi(1)$. Since φ and ψ are positive functionals, $|||\varphi||| = \varphi(1) = \psi(1) = |||\psi|||$. All in all, we conclude that $|||\varphi||| = |||\psi||| = 1$, i.e. $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$. Since $\mu \neq 0$, we also note that $\varphi \neq \psi$. Yet, by definition of mk_{L}, we have $\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\{|\mu(a)| : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\} = 0$. So mk_{L} is not a metric on $\mathscr{S}(\mathfrak{A})$. By contraposition, if $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space, then $\mathrm{cl}(\mathrm{dom}(\mathsf{L})) = \mathfrak{sa}(\mathfrak{A})$. □

Remark 1.1.12. Let $\mathfrak{A}$ be a unital C*-algebra. If L is a seminorm defined on some dense subspace $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$, then mk_{L} is an extended metric in general, as can be checked — in particular, it induces a Hausdorff topology, thanks to the density of the domain of L. It is easy to check that this topology is stronger than the weak* topology (see the proof of Theorem (1.1.18) below for details). So, we could replace the requirement that mk_{L} induces the weak* topology with the requirement that mk_{L} induces a compact topology.

If $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space, then $\mathscr{S}(\mathfrak{A})$ is weak* compact, as an easy corollary of the Alaoglu-Bourbaki theorem [6], since $\mathfrak{A}$ is unital. Hence, the metric space $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}})$ is compact, and thus, in particularly, it has finite diameter.

Notation 1.1.13. The *diameter* $\mathrm{qdiam}(\mathfrak{A}, \mathsf{L})$ of a quantum compact metric space $(\mathfrak{A}, \mathsf{L})$ is the diameter of $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}})$, i.e.

$$\mathrm{qdiam}(\mathfrak{A}, \mathsf{L}) = \sup\left\{\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) : \varphi, \psi \in \mathscr{S}(\mathfrak{A})\right\}.$$

Since mk_{L} is continuous over $\mathscr{S}(\mathfrak{A}) \times \mathscr{S}(\mathfrak{A})$, which is compact, we can in fact write

$$\mathrm{qdiam}(\mathfrak{A}, \mathsf{L}) = \max\left\{\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) : \varphi, \psi \in \mathscr{S}(\mathfrak{A})\right\}.$$

We first establish a simple but very helpful lemma relating the L-seminorm to the norm. We recall from [26, Theorem 4.3.4] that in a C*-algebra, if $a \in \mathfrak{sa}(\mathfrak{A})$, then

$$\|a\|_{\mathfrak{A}} = \max\left\{|\varphi(a)| : \varphi \in \mathscr{S}(\mathfrak{A})\right\}.$$

Lemma 1.1.14. *If* $(\mathfrak{A}, \mathsf{L})$ *is a quantum compact metric space, if* $a \in \operatorname{dom}(\mathsf{L})$*, and if* $\mu \in \mathscr{S}(\mathfrak{A})$*, then*

$$\left\|a - \mu(a)1_{\mathfrak{A}}\right\|_{\mathfrak{A}} \leqslant \mathsf{L}(a)\operatorname{qdiam}(\mathfrak{A}, \mathsf{L}).$$

Proof. Let $\mu \in \mathscr{S}(\mathfrak{A})$ and $a \in \operatorname{dom}(\mathsf{L})$. Let $\varepsilon > 0$. Since $a = a^*$, we thus have

$$\begin{aligned}\left\|a - \mu(a)1_{\mathfrak{A}}\right\|_{\mathfrak{A}} &= \max\left\{|\varphi(a - \mu(a))| : \varphi \in \mathscr{S}(\mathfrak{A})\right\} \\ &= (\mathsf{L}(a) + \varepsilon)\max\left\{\underbrace{\left|\varphi\left(\frac{a}{\mathsf{L}(a)+\varepsilon}\right) - \mu\left(\frac{a}{\mathsf{L}(a)+\varepsilon}\right)\right|}_{\mathsf{L}\left(\frac{a}{\mathsf{L}(a)+\varepsilon}\right)\leqslant 1} : \varphi \in \mathscr{S}(\mathfrak{A})\right\} \\ &\leqslant (\mathsf{L}(a) + \varepsilon)\max\{\mathsf{mk}_{\mathsf{L}}(\varphi, \mu) : \varphi \in \mathscr{S}(\mathfrak{A})\} \leqslant (\mathsf{L}(a) + \varepsilon)\operatorname{qdiam}(\mathfrak{A}, \mathsf{L}).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\left\|a - \mu(a)1_{\mathfrak{A}}\right\|_{\mathfrak{A}} \leqslant \mathsf{L}(a)\operatorname{qdiam}(\mathfrak{A}, \mathsf{L})$, as claimed. □

It follows immediately from Lemma (1.1.14) that a Lipschitz seminorm is zero exactly on the scalars.

Proposition 1.1.15. *If* $(\mathfrak{A}, \mathsf{L})$ *is a quantum compact metric space, then*

$$\{a \in \operatorname{dom}(\mathsf{L}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}.$$

Proof. Let $a \in \operatorname{dom}(\mathsf{L})$ such that $\mathsf{L}(a) = 0$. Fix $\mu \in \mathscr{S}(\mathfrak{A})$. By Lemma (1.1.14), we conclude that

$$\left\|a - \mu(a)1_{\mathfrak{A}}\right\|_{\mathfrak{A}} \leqslant \operatorname{qdiam}(\mathfrak{A}, \mathsf{L})\mathsf{L}(a) = 0,$$

so $a = \mu(a)1_{\mathfrak{A}} \in \mathbb{R}1_{\mathfrak{A}}$, as claimed. □

We can rephrase the definition of the Monge-Kantorovich metric, with sights on our characterization of quantum compact metric spaces.

Lemma 1.1.16. *Let* E *be a vector space, and let* L *be a seminorm on* E*. If* $x \in E$ *such that* $\mathsf{L}(x) = 0$*, then*

$$\forall a \in E \quad \forall t \in \mathbb{R} \quad \mathsf{L}(a + tx) = \mathsf{L}(a).$$

Proof. For all $t \in \mathbb{R}$ and $a \in \operatorname{dom}(\mathsf{L})$, we compute:

$$\begin{aligned}\mathsf{L}(a) &= \mathsf{L}(a + tx - tx) \\ &\leqslant \mathsf{L}(a + tx) + |t|\underbrace{\mathsf{L}(x)}_{=0} = \mathsf{L}(a + tx) \\ &\leqslant \mathsf{L}(a) + |t|\underbrace{\mathsf{L}(x)}_{=0} = \mathsf{L}(a).\end{aligned}$$

This completes our proof. □

Corollary 1.1.17. *Let $\mathfrak{A}$ be a unital C*-algebra. Let L be a seminorm defined on a subspace $\operatorname{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$ such that $\mathsf{L}(1_{\mathfrak{A}}) = 0$. If $\mu \in \mathscr{S}(\mathfrak{A})$, then, for all $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$,*

$$\sup\left\{|\varphi(a) - \psi(a)| : a \in \operatorname{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\right\} \\ = \sup\left\{|\varphi(a) - \psi(a)| : a \in \operatorname{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1, \mu(a) = 0\right\}.$$

Proof. Of course,

$$\sup\left\{|\varphi(a) - \psi(a)| : a \in \operatorname{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1, \mu(a) = 0\right\} \\ \leqslant \sup\left\{|\varphi(a) - \psi(a)| : a \in \operatorname{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\right\}.$$

Now, let $\varepsilon > 0$. There exists $b \in \operatorname{dom}(\mathsf{L})$ such that $\mathsf{L}(b) \leqslant 1$ and

$$|\varphi(b) - \psi(b)| \geqslant \sup\left\{|\varphi(b) - \psi(b)| : a \in \operatorname{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\right\} - \varepsilon.$$

By Lemma (1.1.16), we conclude that $\mathsf{L}(b - \mu(b)1_{\mathfrak{A}}) = \mathsf{L}(b) \leqslant 1$. Of course, $\mu(b - \mu(b)1_{\mathfrak{A}}) = 0$. Moreover, since $\psi(1_{\mathfrak{A}}) = \varphi(1_{\mathfrak{A}}) = 1$, we also note that

$$\left|\varphi(b - \mu(b)) - \psi(b - \mu(b))\right| = |\varphi(b) - \psi(b)|.$$

Therefore, we conclude that

$$\begin{aligned} &\sup\left\{|\varphi(a) - \psi(a)| : a \in \operatorname{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1, \mu(a) = 0\right\} \\ &\geqslant |\varphi(b - \mu(b)1_{\mathfrak{A}}) - \psi(b - \mu(b)1_{\mathfrak{A}})| = |\varphi(b) - \psi(b)| \\ &\geqslant \sup\left\{|\varphi(a) - \psi(a)| : a \in \operatorname{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\right\} - \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, our proof is complete. □

We now establish the core characterization of which seminorms, densely defined on the space of self-adjoint elements of a unital C*-algebra, can be used to metrize the weak* topology on the state space, via Expression (1.1.1).

Theorem 1.1.18. *Let $\mathfrak{A}$ be a unital C*-algebra, and let L be a seminorm defined on a dense subspace $\operatorname{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$ such that $\mathsf{L}(1) = 0$. For all states $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$, we define*

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\left\{\left|\varphi(a) - \psi(a)\right| : a \in \operatorname{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\right\}. \tag{1.1.4}$$

The following assertions are equivalent:

1. *The metric mk_{L} induces the weak* topology on $\mathscr{S}(\mathfrak{A})$.*
2. *For all state $\mu \in \mathscr{S}(\mathfrak{A})$ of $\mathfrak{A}$, the set $\left\{a \in \operatorname{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1, \mu(a) = 0\right\}$ is totally bounded in $\mathfrak{sa}(\mathfrak{A})$.*
3. *There exists a state $\mu \in \mathscr{S}(\mathfrak{A})$ of $\mathfrak{A}$ such that the set $\left\{a \in \operatorname{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1, \mu(a) = 0\right\}$ is totally bounded in $\mathfrak{sa}(\mathfrak{A})$.*

Proof. Assume that mk_{L} metrizes the weak* topology, restricted to $\mathscr{S}(\mathfrak{A})$. If $a \in \mathfrak{sa}(\mathfrak{A})$, we define the following $\mathbb{R}$-valued, continuous, affine function over $\mathscr{S}(\mathfrak{A})$:

$$\widehat{a} : \varphi \in \mathscr{S}(\mathfrak{A}) \mapsto \varphi(a).$$

For any self-adjoint element $a \in \mathfrak{sa}(\mathfrak{A})$ of $\mathfrak{A}$, since $\|a\|_{\mathfrak{A}} = \sup_{\varphi \in \mathscr{S}(\mathfrak{A})} |\varphi(a)|$ by [26, Theorem 4.3.4], we conclude that the function $a \in \mathfrak{sa}(\mathfrak{A}) \mapsto \widehat{a} \in C(\mathscr{S}(\mathfrak{A}))$ is a linear isometry from $\mathfrak{sa}(\mathfrak{A})$.

If $a \in \mathrm{dom}(\mathsf{L}) \subseteq \mathfrak{sa}(\mathfrak{A})$, then, by definition of the metric mk_{L} in Expression (1.1.4), for all $\varphi \in \mathscr{S}(\mathfrak{A})$,

$$\left|\widehat{a}(\varphi) - \widehat{a}(\psi)\right| = |\varphi(a) - \psi(a)| \leqslant \mathsf{L}(a)\mathsf{mk}_{\mathsf{L}}(\varphi, \psi).$$

Therefore, the set $\{\widehat{a} : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$ is equicontinuous over the metric space $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}})$.

Moreover, if $\mu \in \mathscr{S}(\mathfrak{A})$, and if $a \in \mathrm{dom}(\mathsf{L})$ with $\mathsf{L}(a) \leqslant 1$, and $\mu(a) = 0$, we conclude, by Lemma (1.1.14), that

$$\|a\|_{\mathfrak{A}} = \left\|a - \mu(a)1_{\mathfrak{A}}\right\|_{\mathfrak{A}} \leqslant \mathrm{qdiam}(\mathfrak{A}, \mathsf{L})$$

so

$$\left\{\widehat{a} : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1, \mu(a) = 0\right\}$$

is equicontinuous, valued in $[-\mathrm{qdiam}(\mathfrak{A}, \mathsf{L}), \mathrm{qdiam}(\mathfrak{A}, \mathsf{L})]$, over the compact space $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}})$, and thus, by the Arzéla-Ascoli theorem, we conclude that this set is totally bounded in $C(\mathscr{S}(\mathfrak{A}))$. Since $a \in \mathfrak{sa}(\mathfrak{A}) \mapsto \widehat{a}$ is an isometry, we conclude that $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1, \mu(a) = 0\}$ is totally bounded as well.

Of course, Assertion (2) implies Assertion (3).

Last, we assume Assertion (3). Thus, there exists $\mu \in \mathscr{S}(\mathfrak{A})$ such that

$$B := \{a \in \mathrm{dom}(\mathsf{L}) : \mu(a) = 0, \mathsf{L}(a) \leqslant 1\}$$

is totally bounded. Let $(\varphi_j)_{j \in J}$ be a net in $\mathscr{S}(\mathfrak{A})$, weak* converging to φ.

Let $\varepsilon > 0$. Since B is totally bounded, there exists a finite $\frac{\varepsilon}{3}$-dense subset $F \subseteq B$ of B. Specifically, for all $a \in B$, there exists $a' \in F$ such that $\left\|a - a'\right\|_{\mathfrak{A}} < \frac{\varepsilon}{3}$. By definition of weak* convergence, for each $a \in F$, there exists j_a such that, if $j \succ j_a$, then $\left|\varphi_j(a) - \varphi(a)\right| < \frac{\varepsilon}{3}$. Since J is a directed set, and F is finite, there exists $j_F \in J$ such that, for all $a \in F$, we have $j_F \succ j_a$. Therefore, for all $j \succ j_F$, we conclude that $|\varphi_j(a) - \varphi(a)| < \frac{\varepsilon}{3}$. Therefore, for all $a \in B$, if $j \succ j_F$, then there exists $a' \in F$ such that $\left\|a - a'\right\|_{\mathfrak{A}} < \frac{\varepsilon}{3}$, and then

$$\begin{aligned}\left|\varphi_j(a) - \varphi(a)\right| &\leqslant \left|\varphi_j(a) - \varphi_j(a')\right| + \left|\varphi_j(a') - \varphi(a')\right| + \left|\varphi(a') - \varphi(a)\right| \\ &< \left\|a - a'\right\|_{\mathfrak{A}} + \frac{\varepsilon}{3} + \left\|a' - a\right\|_{\mathfrak{A}} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.\end{aligned}$$

Thus, if $j \succ j_F$, then, using Corollary (1.1.17):

$$\begin{aligned}\mathsf{mk}_{\mathsf{L}}(\varphi_j, \varphi) &= \sup\left\{|\varphi_j(a) - \varphi(a)| : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\right\} \\ &= \sup\left\{|\varphi_j(a - \mu(a)1_{\mathfrak{A}}) - \varphi(a - \mu(a)1_{\mathfrak{A}})| : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\right\} \\ &= \sup\left\{|\varphi_j(a) - \varphi(a)| : a \in B\right\} \\ &< \varepsilon.\end{aligned}$$

Therefore, the weak* topology is stronger than the topology induced by mk_{L}.

Now, the topology induced by mk_{L} is, of course, Hausdorff, since mk_{L} is a distance on $\mathscr{S}(\mathfrak{A})$. Therefore, it agrees with the compact weak* topology. □

Theorem 1.1.19. *Let $(\mathfrak{A}, \mathsf{L})$ be an ordered pair with $\mathfrak{A}$ a unital C*-algebra and L is a seminorm defined on a Jordan-Lie subalgebra $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$ such that, for some $K \geqslant 1$:*

$$\forall a, b \in \mathrm{dom}(\mathsf{L}) \quad \max\left\{\mathsf{L}\left(\frac{ab+ba}{2}\right), \mathsf{L}\left(\frac{ab-ba}{2i}\right)\right\} \leqslant K\left(\|a\|_{\mathfrak{A}}\,\mathsf{L}(b) + \mathsf{L}(a)\,\|b\|_{\mathfrak{A}} + \mathsf{L}(a)\mathsf{L}(b)\right),$$

and $\mathsf{L}(1) = 0$.

The following assertions are equivalent.

1. $(\mathfrak{A}, \mathsf{L})$ *is a K-quantum compact metric space.*

2. *The space* $\mathrm{dom}(\mathsf{L})$ *is dense,* $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$, *and there exists a state $\mu \in \mathscr{S}(\mathfrak{A})$ such that*

$$\left\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1, \mu(a) = 0\right\}$$

   is compact in $\mathfrak{A}$.

3. *The space* $\mathrm{dom}(\mathsf{L})$ *is dense,* $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$, *and for all state $\mu \in \mathscr{S}(\mathfrak{A})$, the set*

$$\left\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1, \mu(a) = 0\right\}$$

   is compact in $\mathfrak{A}$.

Proof. If $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space, then $\mathrm{dom}(\mathsf{L})$ is dense by Proposition (1.1.11), and $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) = 0\}$ is reduced to $\mathbb{R}1$, by Proposition (1.1.15).

This theorem then follows from Theorem (1.1.18), and the fact that, in Definition (1.1.1), we require the unit ball of L to be closed. Since $\mathfrak{A}$ is complete, a subset of $\mathfrak{A}$ is compact if, and only if, it is totally bounded and closed in $\mathfrak{A}$. □

We can use Theorem (1.1.19) to establish new examples, starting with the following simple illustration.

Example 1.1.20. Let (X,d) be a compact metric space, and let $\mathfrak{M}$ be a finite dimensional C*-algebra. Let

$$\mathfrak{A} = \left\{ f \in C(X,\mathfrak{M}) : f(x_0) \in C1_{\mathfrak{M}} \right\}.$$

For all $f \in \mathfrak{A}$, we set

$$\mathsf{L}(f) = \sup \left\{ \frac{\left\| f(x) - f(y) \right\|_{\mathfrak{M}}}{d(x,y)} : x,y \in X, x \neq y \right\}.$$

Of course $\mathsf{L}(1_{\mathfrak{A}}) = 0$. By construction, if $\mathsf{L}(f) = 0$, then $f : X \to \mathfrak{M}$ is constant. Since $f(x_0) \in \mathbb{R}1_{\mathfrak{M}}$, we conclude that $f \in \mathbb{R}1_{\mathfrak{A}}$. By Arzéla-Ascoli theorem, $\{f \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(f) \leqslant 1, \mu(f) \leqslant 1\}$ is totally bounded.

We also note that $\mathsf{L}(fg) \leqslant \mathsf{L}(f) \left\| g \right\|_{\mathfrak{A}} + \left\| f \right\|_{\mathfrak{A}} \mathsf{L}(g)$.

Embedd $\mathfrak{M}$ in the algebra $\mathfrak{M}_d$ of $d \times d$ matrices, for d large enough. Let

$$\left\| (a_{jk})_{1 \leqslant j,k \leqslant d} \right\| = \max\{|a_{jk}| : 1 \leqslant j,k \leqslant d\}$$

for all $(a_{jk})_{1 \leqslant j,k \leqslant d} \in \mathfrak{M}_d$. As $\mathfrak{A}$ is finite dimensional, $\|\cdot\|$ is equivalent to $\|\cdot\|_{\mathfrak{M}}$. Let $\left\| f \right\| = \sup_{x \in X} \left\| f(x) \right\|$. Let $f \in \mathfrak{sa}(\mathfrak{A})$. For each f_{jk}, there exists g_{jk} Lipschitz such that $\left\| f_{jk} - g_{jk} \right\| < \varepsilon$. Thus, the Lipschitz functions are dense.

So $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space.

Quantum compact metric spaces can be seen as a generalization of Lipschitz algebras, i.e. algebras of Lipschitz functions over compact metric spaces. Now, Lipschitz algebras are Banach algebras for a natural norm, and a similar result holds for their noncommutative analogues.

Lemma 1.1.21. *Let $\mathfrak{A}$ be a Banach space and let L be a seminorm defined on a dense subspace* $\mathrm{dom}(\mathsf{L})$ *of $\mathfrak{A}$. Let $\|a\|_{\mathrm{dom}(\mathsf{L})} := \|a\|_{\mathfrak{A}} + \mathsf{L}(a)$ for all $a \in \mathrm{dom}(\mathsf{L})$. If $\{a \in \mathfrak{A} : \mathsf{L}(a) \leqslant 1\}$ is closed, then $(\mathrm{dom}(\mathsf{L}), \|\cdot\|_{\mathrm{dom}(\mathsf{L})})$ is a Banach space.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathrm{dom}(\mathsf{L}), \|\cdot\|_{\mathsf{L}})$. Since $\|\cdot\|_{\mathfrak{A}} \leqslant \|\cdot\|_{\mathrm{dom}(\mathsf{L})}$, we conclude that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{sa}(\mathfrak{A})$. Since $(\mathfrak{sa}(\mathfrak{A}), \|\cdot\|_{\mathfrak{A}})$ is complete, there exists $a \in \mathfrak{sa}(\mathfrak{A})$ such that $\lim_{n\to\infty} \|a_n - a\|_{\mathfrak{A}} = 0$.

For all $p,q \in \mathbb{N}$, we have $|\mathsf{L}(a_p) - \mathsf{L}(a_q)| \leqslant \mathsf{L}(a_p - a_q) \leqslant \left\| a_p - a_q \right\|_{\mathrm{dom}(\mathsf{L})}$, so $(\mathsf{L}(a_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, and thus in particular, a bounded sequence, in $\mathrm{dom}(\mathsf{L})$. Let $M > 0$ such that, for all $n \in \mathbb{N}$, we have $\mathsf{L}(a_n) \leqslant M$. Since $\{b \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(b) \leqslant M\}$ is closed, by assumption, in $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}})$, we conclude that $a \in \{b \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(b) \leqslant M\}$, i.e. $\mathsf{L}(a) \leqslant M$.

Now, let $\varepsilon > 0$. Since $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathrm{dom}(\mathsf{L}), \|\cdot\|_{\mathrm{dom}(\mathsf{L})})$, there exists $N \in \mathbb{N}$ such that, for all $p,q \in \mathbb{N}$, if $p \geqslant N$ and $q \geqslant N$, then $\left\| a_p - a_q \right\|_{\mathrm{dom}(\mathsf{L})} < \varepsilon$. In particular, $\mathsf{L}(a_p - a_q) < \varepsilon$. Now, by assumption, $\{b \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(b) \leqslant \varepsilon\}$ is closed in $(\mathfrak{sa}(\mathfrak{A}), \|\cdot\|_{\mathfrak{A}})$. Fix $q \geqslant N$. Since $(a_p - a_q)_{q\in\mathbb{N}}$ converges to $a - a_q$ in $(\mathfrak{sa}(\mathfrak{A}), \|\cdot\|_{\mathfrak{A}})$, we conclude that $\mathsf{L}(a - a_q) \leqslant \varepsilon$. Thus, for all $q \geqslant N$, we have shown $\mathsf{L}(a - a_q) < \varepsilon$, i.e. $\lim_{n\to\infty} \mathsf{L}(a_n - a) = 0$.

Consequently, $\lim_{n\to\infty} \|a - a_n\|_{\mathrm{dom}(\mathsf{L})} = 0$, i.e. $(a_n)_{n\in\mathbb{N}}$ converges to $a \in \mathrm{dom}(\mathsf{L})$. Therefore, $(\mathrm{dom}(\mathsf{L}), \|\cdot\|_{\mathrm{dom}(\mathsf{L})})$ is a Banach space. $\square$

Theorem 1.1.22. *If* $(\mathfrak{A}, \mathsf{L})$ *is a quantum compact metric space, and if, for all* $a \in \mathrm{dom}(\mathsf{L})$*, we define*

$$\|a\|_{\mathrm{dom}(\mathsf{L})} = \|a\|_{\mathfrak{A}} + \mathsf{L}(a),$$

then $(\mathrm{dom}(\mathsf{L}), \|\cdot\|_{\mathrm{dom}(\mathsf{L})})$ *is a Banach Jordan-Lie algebra.*

The canonical injection $a \in \mathrm{dom}(\mathsf{L}) \mapsto a \in \mathfrak{A}$ *is a compact linear map (i.e., it maps the closed unit ball of its domain to a compact subset of* $\mathfrak{sa}(\mathfrak{A})$*).*

Proof. By Lemma (1.1.21), the space $(\mathrm{dom}(\mathsf{L}), \|\cdot\|_{\mathsf{L}})$ is a Banach space.

Let now $C = K(1,1,1,1)$; we recall from Remark (1.1.26) that for all $a, b \in \mathrm{dom}(\mathsf{L})$:

$$\max\{\mathsf{L}(\Re(ab)), \mathsf{L}(\Im(ab))\} \leqslant C(\mathsf{L}(a)\|b\|_{\mathfrak{A}} + \mathsf{L}(b)\|a\|_{\mathfrak{A}} + \mathsf{L}(a)\mathsf{L}(b)).$$

We therefore compute, for all $a, b \in \mathrm{dom}(\mathsf{L})$:

$$\begin{aligned}
\|\Re(ab)\|_{\mathrm{dom}(\mathsf{L})} &= \|\Re(ab)\|_{\mathfrak{A}} + \mathsf{L}(\Re(ab)) \\
&\leqslant \|a\|_{\mathfrak{A}}\|b\|_{\mathfrak{A}} + C(\mathsf{L}(a)\|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}}\mathsf{L}(b) + \mathsf{L}(a)\mathsf{L}(b)) \\
&\leqslant C(\|a\|_{\mathfrak{A}}\|b\|_{\mathfrak{A}} + \mathsf{L}(a)\|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}}\mathsf{L}(b) + \mathsf{L}(a)\mathsf{L}(b)) \\
&\leqslant C(\|a\|_{\mathfrak{A}} + \mathsf{L}(a))(\|b\|_{\mathfrak{A}} + \mathsf{L}(b)) \\
&\leqslant C\|a\|_{\mathrm{dom}(\mathsf{L})}\|b\|_{\mathrm{dom}(\mathsf{L})}.
\end{aligned}$$

Consequently, the bilinear map $(a,b) \in \mathrm{dom}(\mathsf{L})^2 \mapsto \Re ab$ is continuous. A similar argument shows that the Lie product $a, b \in \mathrm{dom}(\mathsf{L}) \mapsto \Im(ab)$ is also continuous. This concludes our proof that $(\mathrm{dom}(\mathsf{L}), \|\cdot\|_{\mathsf{L}})$ is a Banach Jordan-Lie algebra.

It remains to prove that $S := \{a \in \mathrm{dom}(\mathsf{L}) : \|a\|_{\mathsf{L}} \leqslant 1\}$ is compact in $\mathfrak{sa}(\mathfrak{A})$. Let $\mu \in \mathscr{S}(\mathfrak{A})$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in S. The sequence $(\mu(a_n))_{n\in\mathbb{N}}$ is valued in $[-1,1]$, and thus it has a convergent subsequence $(\mu(a_{f(n)})_{n\in\mathbb{N}})$ with limit denoted by l. Now, $\mu(a_{f(n)} - \mu(a_{f(n)})) = 0$ and $\mathsf{L}(a_{f(n)} - \mu(a_{f(n)})) = \mathsf{L}(a_{f(n)}) \leqslant 1$ for all $n \in \mathbb{N}$; since $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space, by Theorem (1.1.19), the set $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1, \mu(a) = 0\}$ is compact in $\mathfrak{sa}(\mathfrak{A})$. Therefore, there exists a convergent subsequence $(a_{f(g(n))} - \mu(a_{f(g(n))}))_{n\in\mathbb{N}}$ of $(a_{f(n)} - \mu(a_{f(n)}))_{n\in\mathbb{N}}$ with limit in $b \in \mathrm{dom}(\mathsf{L})$ such that $\mu(b) = 0$ and $\mathsf{L}(b) \leqslant 1$. Therefore, $(a_{f(g(n))})_{n\in\mathbb{N}}$ converges to $b + l$. By lower semicontinuity of L and continuity of the norm, we conclude $\|b + l\|_{\mathsf{L}} \leqslant 1$, concluding our proof. □

Remark 1.1.23. Let $\mathfrak{A}$ be a unital C*-algebra, and let L be a seminorm defined on a dense subspace of $\mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}(1) = 0$ and $\{a \in \mathrm{dom}(\mathsf{L}) : \|a\|_{\mathfrak{A}} + \mathsf{L}(a) \leqslant 1\}$ a compact set. If $\mathrm{qdiam}(\mathfrak{A}, \mathsf{L}) < \infty$, then mk_{L} metrizes the weak* topology of $\mathscr{S}(\mathfrak{A})$, thus providing another avenue to characterize quantum compact metric spaces. Indeed, let $\mu \in \mathscr{S}(\mathfrak{A})$. Using the same argument as for Lemma (1.1.14), we have $\left\|a - \mu(a)\right\|_{\mathfrak{A}} \leqslant \mathrm{qdiam}(\mathfrak{A}, \mathsf{L})\mathsf{L}(a)$. So

$$\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1, \mu(a) = 0\} \subseteq \{a \in \mathrm{dom}(\mathsf{L}) : \|a\|_{\mathfrak{A}} + \mathsf{L}(a) \leqslant \mathrm{qdiam}(\mathfrak{A}, \mathsf{L}) + 1\}.$$

The set $\{a \in \mathrm{dom}(\mathsf{L}) : \|a\|_{\mathfrak{A}} + \mathsf{L}(a) \leqslant \mathrm{qdiam}(\mathfrak{A}, \mathsf{L}) + 1\}$ is compact (since scaling is continuous) and thus $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1, \mu(a) = 0\}$ is totally bounded. We then apply Theorem (1.1.18).

A recurrent theme is the importance of the domain of the Lipschitz seminorms of quantum compact metric spaces, and a first taste of this is given by the following characterization of finite dimensional quantum compact metric spaces.

Theorem 1.1.24. *Let $\mathfrak{A}$ be a Banach space, and let L be a seminorm defined on a dense subspace $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{A}$ such that $K := \{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) = 0\}$ is finite dimensional, and the image of $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1\}$ in ${}^{\mathfrak{A}}\!/_{K}$ is compact in $\mathfrak{A}$.*

The space $\mathfrak{A}$ is finite dimensional if, and only if, $\mathrm{dom}(\mathsf{L}) = \mathfrak{A}$.

Proof. If $\mathfrak{A}$ is finite dimensional, then the domain $\mathrm{dom}(\mathsf{L})$ of L is also finite dimensional, hence complete, hence closed. Since it is dense in $\mathfrak{A}$, we conclude that $\mathrm{dom}(\mathsf{L}) = \mathfrak{A}$.

Conversely, assume that $\mathrm{dom}(\mathsf{L}) = \mathfrak{A}$. Let $\mathfrak{B} := {}^{\mathfrak{A}}\!/_{K}$, endowed with the quotient norm form $\mathfrak{A}$. Let $\mathfrak{B}_r$ be the image by the canonical surjection of $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant r\}$ in $\mathfrak{B}$, for all $r > 0$. Since $\mathrm{dom}(\mathsf{L}) = \mathfrak{A}$, we observe that $\mathfrak{B} = \bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$. By assumption on L, the set $\mathfrak{B}_n$ is compact, hence closed in $\mathfrak{B}$ for all $n \in \mathbb{N}$ (since $\mathfrak{B}$ is Hausdorff).

Since $\mathfrak{B}$ is complete, by the Baire Category Theorem, there exists $n \in \mathbb{N}$ such that $\mathfrak{B}_n$ has nonempty interior. Let $U \subseteq \mathfrak{B}_n$ be a nonempty open ball. Since $\mathfrak{B}_n$ is compact, U is totally bounded. Therefore, the open unit ball of $\mathfrak{B}$ is totally bounded; as $\mathfrak{B}$ is complete, the closed unit ball of $\mathfrak{B}$ is thus compact. Consequently, $\mathfrak{B}$ is finite dimensional. Therefore, $\mathfrak{A}$ is also finite dimensional as claimed, since ${}^{\mathfrak{A}}\!/_{K} = \mathfrak{B}$ and K are both finite dimensional. □

Corollary 1.1.25. *If $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space, then $\mathfrak{A}$ is finite dimensional if, and only if, $\mathrm{dom}(\mathsf{L}) = \mathfrak{sa}(\mathfrak{A})$.*

Proof. By Theorem (1.1.24), we conclude that $\mathfrak{sa}(\mathfrak{A}) = \mathrm{dom}(\mathsf{L})$ if, and only if, $\mathfrak{sa}(\mathfrak{A})$ is finite dimensional, which in turn is equivalent to $\mathfrak{A}$ being finite dimensional. □

Remark 1.1.26. In the literature, a seemingly more general concept of F-(quasi-)Leibniz inequality was introduced. For any function $F : [0,\infty)^4 \to [0,\infty)$ which is increasing for the product order on $[0,\infty)^4$, let us say that $(\mathfrak{A}, \mathsf{L})$ is a F-quantum compact metric space when it satisfies all the conditions to be a quantum compact metric space in Definition (1.1.1), except that we replace Expression (1.1.2) with:

$$\forall a, b \in \mathrm{dom}(\mathsf{L}) \quad \max\{\mathsf{L}(\Re(ab)), \mathsf{L}(\Im(ab))\} \leqslant F(\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{A}}, \mathsf{L}(a), \mathsf{L}(b)).$$

Let now $C := F(1,1,1,1)$. We then observe that for all $a, b \in \mathrm{dom}(\mathsf{L})$, neither being 0,

$$\begin{aligned}
\mathsf{L}(\Re(ab)) &= (\|a\|_{\mathfrak{A}} + \mathsf{L}(a))\,(\|b\|_{\mathfrak{A}} + \mathsf{L}(b))\,\mathsf{L}\left(\Re\left(\frac{a}{\|a\|_{\mathfrak{A}} + \mathsf{L}(a)}\,\frac{b}{\|b\|_{\mathfrak{A}} + \mathsf{L}(b)}\right)\right) \\
&\leqslant (\|a\|_{\mathfrak{A}} + \mathsf{L}(a))\,(\|b\|_{\mathfrak{A}} + \mathsf{L}(b))\,F(1,1,1,1) \\
&= C(\|a\|_{\mathfrak{A}} + \mathsf{L}(a))(\|b\|_{\mathfrak{A}} + \mathsf{L}(b)) \\
&= C(\|a\|_{\mathfrak{A}}\,\mathsf{L}(b) + \|b\|_{\mathfrak{A}}\,\mathsf{L}(a) + \mathsf{L}(a)\mathsf{L}(b) + \|a\|_{\mathfrak{A}}\,\|b\|_{\mathfrak{A}}).
\end{aligned}$$

Up to replacing a by $a-\mu(a)$ for any state μ of $\mathscr{S}(\mathfrak{A})$, we have $\|a\|_{\mathfrak{A}} \leqslant \mathrm{qdiam}(\mathfrak{A},\mathsf{L})\mathsf{L}(a)$ by Lemma (1.1.14), which still applies still Equation (1.1.2) of Definition (1.1.1) played no role in its proof. So we conclude that for all $a,b \in \mathrm{dom}(\mathsf{L})$,

$$\mathsf{L}(\Re(ab)) \leqslant K(\mathsf{L}(a)\|b\|_{\mathfrak{A}} + \mathsf{L}(b)\|a\|_{\mathfrak{A}} + \mathsf{L}(a)\mathsf{L}(b)),$$

where $K = \max\{C, C\mathrm{qdiam}(\mathfrak{A},\mathsf{L})\}$. Note that in particular, for all $a \in \mathrm{dom}(\mathsf{L})$, we have $\mathsf{L}(a) \leqslant K\mathsf{L}(a)$ by choosing $b = 1_{\mathfrak{A}}$, so $K \geqslant 1$ (or, when $\mathfrak{A} = \mathbb{C}$, K may as well be assumed to be greater than 1!). Thus, $(\mathfrak{A},\mathsf{L})$ is a K-quantum compact metric space in the sense of Definition (1.1.1).

Owing to their current relative importance in the field, Leibniz quantum compact metric spaces, which are more restrictive that 1-quantum compact metric spaces, and more generally, (C,D)-quantum compact metric spaces, which would be F-quantum compact metric spaces for the function $F : x, y, l_x, l_y \mapsto C(xl_y + yl_x) + Dl_x l_y$, are given a special name, as they form a subclass of the $\max\{C,D\}$-quantum compact metric spaces. Whether this distinction is actually relevant is not entirely clear. We will however see that it may be desirable to restrict our attention to subclasses of quantum compact metric spaces with various additional properties, and this will be relevant to the construction of the propinquity.

1.2 APPROXIMATELY FINITE DIMENSIONAL C*-ALGEBRAS AS QUANTUM COMPACT METRIC SPACES

AF algebras provide interesting examples of quantum compact metric spaces. There are different approaches, and we take the simplest one here. We begin with a few facts about conditional expectations.

Definition 1.2.1. A *conditional expectation* $\mathbb{E}(\cdot|\mathfrak{B}) : \mathfrak{A} \to \mathfrak{B}$ onto $\mathfrak{B}$, where $\mathfrak{A}$ is a C*-algebra and $\mathfrak{B}$ is a C*-subalgebra of $\mathfrak{A}$, is a linear positive map of norm 1 such that for all $b,c \in \mathfrak{B}$ and $a \in \mathfrak{A}$ we have:

$$\mathbb{E}(bac|\mathfrak{B}) = b\mathbb{E}(a|\mathfrak{B})c.$$

Lemma 1.2.2. *Let $\mathfrak{A}$ be a C*-algebra and $\mathfrak{B} \subseteq \mathfrak{A}$ be a C*-subalgebra of $\mathfrak{A}$. If $\mathbb{E}(\cdot|\mathfrak{B}) : \mathfrak{A} \mapsto \mathfrak{B}$ is a conditional expectation onto $\mathfrak{B}$, then the seminorm:*

$$\mathsf{S} : a \in \mathfrak{A} \mapsto \|a - \mathbb{E}(a|\mathfrak{B})\|_{\mathfrak{A}}$$

is a $(2,0)$-quasi-Leibniz seminorm.

Proof. Let $a, b \in \mathfrak{A}$. We have:

$$\begin{aligned}
\mathrm{S}(ab) &= \|ab - \mathbb{E}(ab|\mathfrak{B})\|_{\mathfrak{A}} \\
&\leqslant \|ab - a\mathbb{E}(b|\mathfrak{B})\|_{\mathfrak{A}} + \|a\mathbb{E}(b|\mathfrak{B}) - \mathbb{E}(ab|\mathfrak{B})\|_{\mathfrak{A}} \\
&\leqslant \|a\|_{\mathfrak{A}}\|b - \mathbb{E}(b|\mathfrak{B})\|_{\mathfrak{A}} \\
&\quad + \|a\mathbb{E}(b|\mathfrak{B}) - \mathbb{E}(a\mathbb{E}(b|\mathfrak{B})|\mathfrak{B}) + \mathbb{E}(a(\mathbb{E}(b|\mathfrak{B}) - b)|\mathfrak{B})\|_{\mathfrak{A}} \\
&\leqslant \|a\|_{\mathfrak{A}}\|b - \mathbb{E}(b|\mathfrak{B})\|_{\mathfrak{A}} + \|a - \mathbb{E}(a|\mathfrak{B}))\|_{\mathfrak{A}}\|\mathbb{E}(b|\mathfrak{B})\|_{\mathfrak{A}} \\
&\quad + \|\mathbb{E}(a(b - \mathbb{E}(b|\mathfrak{B}))|\mathfrak{B})\|_{\mathfrak{A}} \\
&\leqslant \|a\|_{\mathfrak{A}}\|b - \mathbb{E}(b|\mathfrak{B})\|_{\mathfrak{A}} + \|a - \mathbb{E}(a|\mathfrak{B})\|_{\mathfrak{A}}\|\mathbb{E}(b|\mathfrak{B})\|_{\mathfrak{A}} \\
&\quad + \|a\|_{\mathfrak{A}}\|b - \mathbb{E}(b|\mathfrak{B})\|_{\mathfrak{A}} \\
&\leqslant 2\|a\|_{\mathfrak{A}}\|b - \mathbb{E}(b|\mathfrak{B})\|_{\mathfrak{A}} + \|a - \mathbb{E}(a|\mathfrak{B}))\|_{\mathfrak{A}}\|b\|_{\mathfrak{A}} \\
&\leqslant 2\left(\|a\|_{\mathfrak{A}}\mathrm{S}(b) + \|b\|_{\mathfrak{A}}\mathrm{S}(a)\right).
\end{aligned}$$

This proves our lemma. □

AF algebras with a faithful tracial state provide us with useful conditional expectations, for our purpose.

Theorem 1.2.3. *If $\mathfrak{A} = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}}\mathfrak{A}_n\right)$ is a unital C*-algebra with a faithful tracial state τ, where $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ is an increasing sequence of C*-subalgebras of $\mathfrak{A}$, with $\mathbb{C}1 = \mathfrak{A}_0$, then for all $n \in \mathbb{N}$, there exists a unique conditional expectation $\mathbb{E}_n : \mathfrak{A} \twoheadrightarrow \mathfrak{A}_n$ onto $\mathfrak{A}_n$ such that $\tau \circ \mathbb{E}_n = \tau$.*

Proof. Let π be the GNS representation of $\mathfrak{A}$ on the Hilbert space $L^2(\mathfrak{A}, \tau)$ (the completion of $\mathfrak{A}$ for $a, b \in \mathfrak{A} \mapsto \tau(a^* b)$).

To begin with, we note that, from the standard GNS construction, we have the following:

1. since τ is faithful, the map $\xi : a \in \mathfrak{A} \mapsto a \in L^2(\mathfrak{A}, \tau)$ is injective (it could also be written $a \in \mathfrak{A} \mapsto a\omega$ where $\omega = \xi(1)$ is the canonical cyclic vector),

2. since $\|\xi(a)\|_{L^2(\mathfrak{A},\mu)} = \sqrt{\tau(a^* a)} \leqslant \|a\|_{\mathfrak{A}}$ for all $a \in \mathfrak{A}$, the map ξ is a continuous (weak) contraction,

3. by construction, $\xi(ab) = \pi(a)\xi(b)$ for all $a, b \in \mathfrak{A}$.

Let $n \in \mathbb{N}$. Since $\mathfrak{A}_n$ is finite dimension, $\xi(\mathfrak{A}_n)$ is a closed subspace of $L^2(\mathfrak{A}, \tau)$. Let P_n be the orthogonal projection from $L^2(\mathfrak{A}, \tau)$ onto $\xi(\mathfrak{A}_n)$.

We thus note that for all $a \in \mathfrak{A}$, we have $P_n(\xi(a)) \in \xi(\mathfrak{A}_n)$, thus, since ξ is injective, there exists a unique $\mathbb{E}_n(a) \in \mathfrak{A}_n$ with $\xi(\mathbb{E}_n(a)) = P_n(\xi(a))$.

If $a \in \mathfrak{A}_n$, then $P_n\xi((a)) = \xi(a)$ so $\mathbb{E}_n(a) = a$. Thus $\mathbb{E}_n$ is onto $\mathfrak{A}_n$, and restricts to the identity on $\mathfrak{A}_n$.

We now prove that P_n commutes with $\pi(a)$ for all $a\mathfrak{A}_n$. Let $a \in \mathfrak{A}_n$. We note that if $b \in \mathfrak{A}_n$ then $\pi(a)\xi(b) = \xi(ab) \in \xi(\mathfrak{A}_n)$ since $\mathfrak{A}_n$ is a subalgebra of $\mathfrak{A}$. Thus

$\pi(a)\xi(\mathfrak{A}_n) \subseteq \xi(\mathfrak{A}_n)$. Since $\mathfrak{A}_n$ is closed under the adjoint operation, and π is a *-representation, we have $\pi(a^*)\xi(\mathfrak{A}_n) \subseteq \xi(\mathfrak{A}_n)$. Thus, if we let $x \in \xi(\mathfrak{A}_n)^\perp$ and $y \in \xi(\mathfrak{A}_n)$, we then have:

$$\langle \pi(a)x, y\rangle_= \langle x, \pi(a^*)y\rangle_= 0,$$

i.e. $\pi(a)\xi(\mathfrak{A}_n))^\perp \subseteq \xi(\mathfrak{A}_n)^\perp$. Consequently, if $x \in L^2(\mathfrak{A},\tau)$, writing $x = P_n x + P_n^\perp x$, we have:

$$P_n\pi(a)x = P_n\pi(a)P_n x + P_n\pi(a)P_n^\perp x = \pi(a)P_n x.$$

In other words, P_n commutes with $\pi(a)$ for all $a \in \mathfrak{A}_n$.

As a consequence, for all $a \in \mathfrak{A}_n$ and $b \in \mathfrak{A}$:

$$\xi(\mathbb{E}_n(ab)) = P_n\pi(a)\xi(b) = \pi(a)P_n\xi(b) = \pi(a)\xi(\mathbb{E}_n(b)) = \xi(a\mathbb{E}_n(b)).$$

Thus $\mathbb{E}_n(ab) = a\mathbb{E}_n(b)$ for all $a \in \mathfrak{A}_n$ and $b \in \mathfrak{A}$.

We now wish to prove that $\mathbb{E}_n$ is a *-linear map. Let $J : \xi(x) \mapsto \xi(x^*)$. The key observation is that, since τ is a trace:

$$\langle J\xi(x), J\xi(y)\rangle = \tau(yx^*) = \tau(x^*y) = \langle x, y\rangle$$

hence J is an conjugate-linear isometry and can be extended to $L^2(\mathfrak{A},\tau)$. It is easy to check that J is surjective, as it has a dense range and is isometric, in fact $J = J^* = J^{-1}$. This is the only point where we use that τ is a trace.

We now check that P_n and J commute. To begin with, we note that:

$$(JP_nJ)(JP_nJ) = JP_nJ$$

and thus the self-adjoint operator JP_nJ is a projection. Let $a \in \mathfrak{A}$. Then:

$$JP_nJ\xi(a) = JP_n\xi(a^*) = J\xi(\mathbb{E}_n(a^*)) = \xi(\mathbb{E}_n(a^*)^*) \in \xi(\mathfrak{A}_n).$$

Thus $JP_nJ = P_n$, so P_n and J commute since $J^2 = 1_{\mathfrak{B}(L^2(\mathfrak{A},\tau))}$.

Consequently for all $a \in \mathfrak{A}$:

$$\xi(\mathbb{E}_n(a^*)) = P_n\xi(a^*) = P_nJ\xi(a) = JP_n\xi(a) = J\xi(\mathbb{E}_n(a)) = \xi(\mathbb{E}_n(a)^*),$$

so $\mathbb{E}_n(a^*) = \mathbb{E}_n(a)^*$.

In particular, we note that for all $a \in \mathfrak{A}$ and $b, c \in \mathfrak{A}_n$ we have:

$$\mathbb{E}_n(bac) = b\mathbb{E}_n(ac) = b\mathbb{E}_n(c^*a^*)^* = b(c^*\mathbb{E}_n(a)^*)^*) = b\mathbb{E}_n(a)c.$$

To prove that $\mathbb{E}_n$ is a positive map, we begin by checking that it preserves the state τ. First note that $1_\mathfrak{A} \in \mathfrak{A}_n$ so $\omega \in \xi(\mathfrak{A}_n)$, and thus $P_n\omega = \omega$. Thus for all $a \in \mathfrak{A}$:

$$\begin{aligned}
\tau(\mathbb{E}_n(a)) &= \langle \pi(\mathbb{E}_n(a))\omega, \omega\rangle \\
&= \langle \xi(\mathbb{E}_n(a)), \omega\rangle_= \langle P_n\xi(a), \omega\rangle \\
&= \langle \xi(a), P_n\omega\rangle_= \langle \pi(a)\omega, P_n\omega\rangle \\
&= \langle \pi(a)\omega, \omega\rangle = \tau(a).
\end{aligned}$$

Thus $\mathbb{E}_n$ preserves the state τ. More generally, using the conditional expectation property, for all $b, c \in \mathfrak{A}_n$ and $a \in \mathfrak{A}$:

$$\tau(b\mathbb{E}_n(a)c) = \tau(bac).$$

We now prove that $\mathbb{E}_n$ is positive. First, τ restricts to a faithful state of $\mathfrak{A}_n$ and $L^2(\mathfrak{A}_n, \tau)$ is given canonically by $\xi(\mathfrak{A}_n)$. Let now $a \in \mathfrak{sa}(\mathfrak{A})$ with $a \geqslant 0$. We now have for all $b \in \mathfrak{A}_n$ that:

$$\langle \mathbb{E}_n(a)\xi(b), \xi(b)\rangle = \tau(b^* \mathbb{E}_n(a)b) = \tau(b^* ab) \geqslant 0.$$

Thus the operator $\mathbb{E}_n(a)$ is positive in $\mathfrak{A}_n$. Thus $\mathbb{E}_n$ is positive.

Since $\mathbb{E}_n$ restricts to the identity on $\mathfrak{A}_n$, this map is of norm at least one. Now, let $a \in \mathfrak{sa}(\mathfrak{A})$ and $\varphi \in \mathscr{S}(\mathfrak{A})$. Then $\varphi \circ \mathbb{E}_n$ is a state of $\mathfrak{A}$ since $\mathbb{E}_n$ is positive and unital. Thus $\left|\varphi \circ \mathbb{E}_n(a)\right| \leqslant \|a\|_{\mathfrak{A}}$. As $\mathbb{E}_n(\mathfrak{sa}(\mathfrak{A})) \subseteq \mathfrak{sa}(\mathfrak{A})$, we have:

$$\forall a \in \mathfrak{sa}(\mathfrak{A}) \quad \|\mathbb{E}_n(a)\|_{\mathfrak{A}} = \sup\left\{|\varphi \circ \mathbb{E}_n(a)| : \varphi \in \mathscr{S}(\mathfrak{A})\right\} \leqslant \|a\|_{\mathfrak{A}}. \tag{1.2.1}$$

Thus $\mathbb{E}_n$ restricted to $\mathfrak{sa}(\mathfrak{A})$ is a linear map of norm 1.

On the other hand, for all $a \in \mathfrak{A}$, we have:

$$\begin{aligned}
0 &\leqslant \mathbb{E}_n\left((a - \mathbb{E}_n(a))^* (a - \mathbb{E}_n(a))\right) \\
&= \mathbb{E}_n\left(a^* a\right) - \mathbb{E}_n\left(\mathbb{E}_n(a)^* a\right) - \mathbb{E}_n\left(a^* \mathbb{E}_n(a)\right) + \mathbb{E}_n\left(\mathbb{E}_n(a)^* \mathbb{E}_n(a)\right) \\
&= \mathbb{E}_n\left(a^* a\right) - \mathbb{E}_n(a)^* \mathbb{E}_n(a).
\end{aligned}$$

Thus for all $a \in \mathfrak{A}$ we have:

$$\begin{aligned}
\|\mathbb{E}_n(a)\|_{\mathfrak{A}}^2 &= \left\|\mathbb{E}_n(a)^* \mathbb{E}_n(a)\right\|_{\mathfrak{A}} \\
&\leqslant \left\|\mathbb{E}_n(a^* a)\right\|_{\mathfrak{A}} \\
&\leqslant \left\|a^* a\right\|_{\mathfrak{A}} = \|a\|_{\mathfrak{A}}^2 \text{ by Inequality (1.2.1).}
\end{aligned}$$

Thus $\mathbb{E}_n$ has norm 1. We conclude that $\mathbb{E}_n$ is a conditional expectation onto $\mathfrak{A}_n$ which preserves τ.

Now, assume $T : \mathfrak{A} \to \mathfrak{A}_n$ is a unital conditional expectation such that $\tau \circ \mathbb{E}_n = \tau$. As before, we have:

$$\tau(bT(a)c) = \tau(bac)$$

for all $a \in \mathfrak{A}$ and $b, c \in \mathfrak{A}_n$. Thus, for all $x, y \in L^2(\mathfrak{A}_n, \tau)$ and for all $a \in \mathfrak{A}$, we compute:

$$\langle T(a)x, y\rangle = \tau(y^* T(a)x) = \tau(y^* ax) = \tau(y^* \mathbb{E}_n(a)x) = \langle \mathbb{E}_n(a)x, y\rangle$$

and thus $\mathbb{E}_n(a) = T(a)$ for all $a \in \mathfrak{A}$. So $\mathbb{E}_n$ is the unique conditional expectation from $\mathfrak{A}$ onto $\mathfrak{A}_n$ which preserves τ. □

With this in mind, we can introduce an interesting quantum compact metric space structure on many AF algebras.

Theorem 1.2.4. *Let $\mathfrak{A} = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right)$ be a unital C*-algebra, where $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ is an increasing sequence of C*-subalgebras of $\mathfrak{A}$, with $\mathbb{C}1 = \mathfrak{A}_0$. If $\mathfrak{A}$ has a tracial state τ, then there exists a (unique) conditional expectation $\mathbb{E}_n : \mathfrak{A} \to \mathfrak{A}_n$ on each $\mathfrak{A}_n$ such that $\tau \circ \mathbb{E}_n = \mathbb{E}_n$, for each $n \in \mathbb{N}$; moreover, if $(d_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence on nonnegative real numbers with $\lim_{n\to\infty} d_n = \infty$, then, setting for all $a \in \mathfrak{sa}(\mathfrak{A})$:*

$$\mathsf{L}(a) := \sup\left\{d_n \|a - \mathbb{E}_n(a)\|_: n \in \mathbb{N}\right\},$$

allowing for infinity, then $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space.

Proof. First, we observe that if $\mathsf{L}(a) = 0$, then $a \in \mathbb{R}$: If $\mathsf{L}(a) = 0$, then $\|a - \mathbb{E}_0(a)\|_{\mathfrak{A}} = 0$. Now, $\mathbb{E}_0 = \tau$ since $\tau \circ \mathbb{E}_0 = \tau$, and $\mathfrak{A}_0 = \mathbb{C}$. So $a \in \mathfrak{sa}(\mathfrak{A}) \cap \mathbb{C}1 = \mathbb{R}1$.

Now, for all $n \in \mathbb{N}$, we have $\mathfrak{sa}(\mathfrak{A}_n) \subseteq \mathrm{dom}(\mathsf{L}_n)$. In particular, $\mathrm{dom}(\mathsf{L})$ is dense in $\mathfrak{sa}(\mathfrak{A})$. It is immediate since if $a \in \mathfrak{A}_n$, then $\mathbb{E}_m(a) = a$ for all $m \geqslant n$.

We already know that L satisfies the $(2,0)$-Leibniz inequality, has dense domain, and its kernel is $\mathbb{R}$.

Let $B = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leqslant 1, \tau(a) = 0\}$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that, for all $n \geqslant N$, we have $d_n > \frac{2}{\varepsilon}$. If $a \in B$, then $\mathbb{E}_N(a) \in \mathfrak{A}_N$, and $\tau \circ \mathbb{E}_N(a) = \tau(a) = 0$. So $\mathbb{E}(B) \subseteq C := \{b \in \mathfrak{sa}(\mathfrak{A}_N) : \mathsf{L}(b) \leqslant 1, \tau(b) = 0\}$. On the other hand, by definition of L, we also have $\|a - \mathbb{E}_N(a)\|_{\mathfrak{A}} \leqslant d_n < \frac{\varepsilon}{2}$.

Now, let $\mathfrak{B} := \ker\tau \cap \mathfrak{sa}(\mathfrak{A}_N)$. Of course, $\mathfrak{B}$ is finite dimensional, and moreover, the restriction of L to $\mathfrak{B}$ is now a norm (note that it is defined on all of $\mathfrak{sa}(\mathfrak{A}_N)$ by construction). So C is the closed unit ball for some norm in finite dimension, and thus it is compact. So C is also totally bounded for the C*-norm on $\mathfrak{A}_N$, since all norms are equivalent in finite dimension.

Therefore, there exists a finite $\frac{\varepsilon}{2}$-dense subset of C. Therefore, for all $a \in B$, there exists $c \in F$ such that $\|a - c\|_{\mathfrak{A}} \leqslant \|a - \mathbb{E}_N(a)\|_{\mathfrak{A}} + \|\mathbb{E}_N(a) - b\|_{\mathfrak{A}} < \varepsilon$. Thus, B is totally bounded. By Theorem (1.1.19), we conclude $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space. $\square$

1.3 QUANTUM COMPACT METRIC SPACES FROM ERGODIC ACTIONS OF METRIC COMPACT GROUPS

Definition 1.3.1. A *length function* $\ell : G \to \mathbb{R}$ over a group G is a real-valued function such that

1. $\{g \in G : \ell(g) = 0\} = \{e\}$ where e is the unit of G,
2. $\ell(gg') \leqslant \ell(g) + \ell(g')$ for all $g, g' \in G$,
3. $\ell(g^{-1}) = \ell(g)$ for all $g \in G$.

We immediately note that $0 = \ell(e) \leqslant \ell(g) + \ell(g^{-1}) = 2\ell(g)$ so a length function is always valued in $[0,\infty)$. If G is a group, and if d is a left invariant metric on G, then setting $\ell(g) := d(e, g)$ for all $g \in G$ defines a length function over G. Conversely, if ℓ is a length function over G, then setting $d : g, h \in G \mapsto \ell(h^{-1}g)$ defines a left invariant metric on G.

We will prove, in this section, the following theorem, which provides us with many core examples for the theory of quantum compact metric spaces.

Theorem 1.3.2. *Let α be a strongly continuous action of a compact group G on a unital C*-algebra $\mathfrak{A}$. Let ℓ be a continuous length function over G. For all $a \in \mathfrak{sa}(\mathfrak{A})$, we define:*

$$\mathsf{L}(a) := \sup\left\{ \frac{\left\|a - \alpha^g(a)\right\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{e\} \right\}, \tag{1.3.1}$$

allowing for the value ∞.

The ordered pair $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space if, and only if,

$$\left\{a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a\right\} = \mathbb{C}.$$

Of course, L defined in Theorem (1.3.2) is a seminorm, defined on some subspace of $\mathfrak{sa}(\mathfrak{A})$ (at least on $\mathbb{R}1$!). We will spend the rest of this section proving this result and illustrating it with some specific examples. We begin with the easy observation that our condition is necessary.

Necessary condition. If $a \in \mathfrak{A}$ is chosen so that $\alpha^g(a) = a$ for all $g \in G$, then $\alpha^g(\Re a) = \frac{1}{2}\left(\alpha^g(a) + \alpha^g(a)^*\right) = \frac{1}{2}(a + a^*) = \Re a$. Now, $\Re a \in \mathfrak{sa}(\mathfrak{A})$ and by definition, $\mathsf{L}(\Re a) = 0$. Since $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space, we conclude that $\Re a \in \mathbb{R}$. We similarly conclude that $\Im a \in \mathbb{R}$, and thus $a \in \mathbb{C}$, as claimed. □

The sufficient condition is more involved. We begin with the easier observations. Let us fix our notation for the following few results. We let G be a compact group, endowed with a continuous length function ℓ. We assume given a unital C*-algebra $\mathfrak{A}$, and a strongly continuous action α of G on $\mathfrak{A}$, with the property that its fixed point C*-subalgebra $\mathfrak{A}_1 := \{a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a\}$ is $\mathbb{C}$. We define L on $\mathfrak{A}$ as in Expression (1.3.1) of Theorem (1.3.2).

Lemma 1.3.3. *We have $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}$.*

Proof. If $\mathsf{L}(a) = 0$ then $\left\|\alpha^g(a) - a\right\|_{\ell(g)} = 0$ for all $g \in G \setminus \{e\}$, and thus $\alpha^g(a) = a$. So $a \in \mathfrak{sa}(\mathfrak{A}) \cap \mathbb{C}1 = \mathbb{R}$. □

Lemma 1.3.4. *For all $a, b \in \mathfrak{sa}(\mathfrak{A})$,*

$$\max\{\mathsf{L}(\Re ab), \mathsf{L}(\Im ab)\} \leqslant \|a\|_{\mathfrak{A}} \mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{A}}.$$

Proof. We extend L to a seminorm on $\mathfrak{A}$ (allowing for the value ∞) by setting, for all $a \in \mathfrak{A}$:

$$\mathsf{L}(a) := \sup\left\{ \frac{\left\|a - \alpha^g(a)\right\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{e\} \right\}.$$

Let $a, b \in \mathfrak{A}$. Since, for all $g \in G$:

$$\begin{aligned}
\left\| ab - \alpha^g(ab) \right\|_{\mathfrak{A}} &= \left\| ab - \alpha^g(a)\alpha^g(b) \right\|_{\mathfrak{A}} \\
&\leqslant \left\| a(b - \alpha^b(b)) \right\|_{\mathfrak{A}} + \left\| (a - \alpha^g(a))\alpha^g(b) \right\|_{\mathfrak{A}} \\
&\leqslant \|a\|_{\mathfrak{A}} \left\| b - \alpha^g(b) \right\|_{\mathfrak{A}} + \left\| a - \alpha^g(a) \right\|_{\mathfrak{A}} \left\| \alpha^g(b) \right\|_{\mathfrak{A}} \\
&= \|a\|_{\mathfrak{A}} \left\| b - \alpha^g(b) \right\|_{\mathfrak{A}} + \left\| a - \alpha^g(a) \right\|_{\mathfrak{A}} \|b\|_{\mathfrak{A}},
\end{aligned}$$

we conclude that $\mathsf{L}(ab) \leqslant \|a\|_{\mathfrak{A}} \mathsf{L}(b) + \mathsf{L}(a) \|b\|_{\mathfrak{A}}$.

Therefore, since L is a seminorm, for any $a, b \in \mathfrak{sa}(\mathfrak{A})$:

$$\begin{aligned}
\mathsf{L}(\Re ab) &\leqslant \frac{1}{2} (\mathsf{L}(ab) + \mathsf{L}(ba)) \\
&\leqslant \frac{1}{2} (\|a\|_{\mathfrak{A}} \mathsf{L}(b) + \mathsf{L}(a) \|b\|_{\mathfrak{A}} + \|b\|_{\mathfrak{A}} \mathsf{L}(a) + \mathsf{L}(b) \|a\|_{\mathfrak{A}}) \\
&= \|a\|_{\mathfrak{A}} \mathsf{L}(b) + \mathsf{L}(a) \|b\|_{\mathfrak{A}}.
\end{aligned}$$

A similar reasoning applies for $\Im ab$ in place of $\Re ab$, thus concluding our proof. □

Remark 1.3.5. If L is densely defined on the self-adjoint space $\mathfrak{sa}(\mathfrak{A})$ of a unital C*-algebra $\mathfrak{A}$, and is obtained as the restriction of some seminorm densely defined on $\mathfrak{A}$ which satisfies the usual Leibniz inequality, then the proof of Lemma (1.3.4) applies to prove L satisfies our form of the Leibniz inequality.

Lemma 1.3.6. *L is lower semicontinuous over $\mathfrak{sa}(\mathfrak{A})$.*

Proof. For each $g \in G \setminus \{e\}$, the function $a \in \mathfrak{A} \mapsto \frac{\|a - \alpha^g(a)\|_{\mathfrak{A}}}{\ell(g)}$ is continuous. Thus L, as the pointwise supremum of continuous functions over $\mathfrak{sa}(\mathfrak{A})$, is lower semicontinuous. □

In order to obtain the other properties of L, we will work with the following generalized convolution operator.

Notation 1.3.7. Let λ be the Haar probability measure on G. For all $f \in L^1(G, \lambda)$, we define

$$\alpha^f \left| \begin{array}{ccc} \mathfrak{A} & \mapsto & \mathfrak{A} \\ a & \longmapsto & \int_G f(g)\alpha^g(a)\, d\lambda(g) \end{array} \right.$$

We recall that there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $C(G)$ such that, for all $h \in C(G)$, we have $\lim_{n \to \infty} \int_G f_n h \, d\lambda = h(0)$; of course, if $A \subseteq C(G)$ is a dense subspace, then we can ask for $f_n \in A$ for all $n \in \mathbb{N}$. With this in mind, we prove the following.

Lemma 1.3.8. $\operatorname{dom}(\mathsf{L})$ *is dense in* $\mathfrak{sa}(\mathfrak{A})$.

Proof. For any $f \in C(G)$, we observe:

$$\begin{aligned}\alpha^f(a) - \alpha^g(\alpha^f(a)) &= \int_G f(h)\alpha^h(a)\, d\lambda(h) - \int_G f(h)\alpha^{gh}(a)\, d\lambda(h) \\ &= \int_G f(g)\alpha^h(a)\, d\lambda(h) - \int_G f(g^{-1}h)\alpha^h(a)\, d\lambda(h) \\ &= \int_G (f(h) - f(g^{-1}h))\alpha^h(a)\, d\lambda(h),\end{aligned}$$

so:

$$\begin{aligned}\frac{\left\|\alpha^f(a) - \alpha^g(\alpha^f(a))\right\|_{\mathfrak{A}}}{\ell(g)} &= \int_G \frac{|f(h) - f(g^{-1}h)}{\ell(g)} \left\|\alpha^h(a)\right\|_{\mathfrak{A}} d\lambda(h)\, \|a\|_{\mathfrak{A}} \\ &\leqslant \int_G \frac{|f(h) - f(g^{-1}h)}{\ell(g)}\, d\lambda(h)\, \|a\|_{\mathfrak{A}} \\ &= \int_G \frac{|f(h) - f(g^{-1}h)}{\ell(g^{-1})}\, d\lambda(h)\, \|a\|_{\mathfrak{A}}.\end{aligned}$$

Let E be the space of all functions $f \in C(G)$ for which there exists $K_f > 0$ such that $\sup_{g\in G}\int_G \left|f(h) - f(g^{-1}h)\right| d\lambda(h) \leqslant K_f\,\ell(g)$. First, note that all the constant functions belong to E. Moreover, note that since $|\ell(hk) - \ell(g^{-1}hk)| \leqslant \ell(g^{-1}) = \ell(g)$ for all $g, h, k \in G$, the functions $h \in G \mapsto \ell(hk)$ all belong to E; therefore E separates the points of G (since $\ell(hk^{-1}) = 0$ if, and only if, $h = k$). Last, let $u, v \in E$. We note that, for all $g \in G$:

$$\begin{aligned}\int_G |uv(h) - uv(g^{-1}h)|\, d\lambda(h) \leqslant \|u\|_{C(G)} \int_G |v(h) - v(g^{-1}h)|\, d\lambda(h) \\ + \int_G |u(h) - u(g^{-1}h)|\, d\lambda(h)\, \|v\|_{C(G)},\end{aligned}$$

we also have $uv \in E$. So E is a subalgebra of $C(G)$ which separates the points and include the unit of $C(G)$, and thus it is dense in $C(G)$.

Therefore, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in E such that, for all $u \in C(G)$, we have $\lim_{n\to\infty}\int_G f\,u\,d\lambda = u(e)$; we also can assume that $f_n \geqslant 0$ and that $\int f_n\,d\lambda = 1$ for all $n \in \mathbb{N}$. Therefore, for all $n \in \mathbb{N}$:

$$\begin{aligned}\left\|a - \alpha^{f_n}(a)\right\|_{\mathfrak{A}} &= \left\|\int_G f_n(h)a - f_n(h)\alpha^h(a)\, d\lambda(h)\right\|_{\mathfrak{A}} \\ &\leqslant \int_G f_n(h)\left\|a - \alpha^h(a)\right\|_{\mathfrak{A}} d\lambda(h) \xrightarrow{n\to\infty} \left\|a - \alpha^e(a)\right\|_{\mathfrak{A}} = 0.\end{aligned}$$

Consequently, $(\alpha^{f_n}(a))_{n\in\mathbb{N}}$ converges to a, and $\alpha^{f_n}(a) \in \operatorname{dom}(\mathsf{L})$ for all $n \in \mathbb{N}$. So $\operatorname{dom}(\mathsf{L})$ is dense in $\mathfrak{sa}(\mathfrak{A})$. □

The operator α^f in Notation (1.3.7) enjoys two more relevant properties. First, it has a nice property related to our Lip-norm which is sort of mean-value theorem.

Lemma 1.3.9. *If $f \in L^1(G,\lambda)$, then for all $a \in \mathfrak{A}$,*

$$\left\|a - \alpha^f(a)\right\|_{\mathfrak{A}} \leqslant \mathsf{L}(a)\,\|f\|_{L^1(G,\lambda)}.$$

Proof. We compute:

$$\begin{aligned}\left\|\alpha^f(a)-a\right\|_{\mathfrak{A}} &= \left\|\int_G f(g)\left(\alpha^g(a)-a\right)d\lambda(g)\right\|_{\mathfrak{A}} \text{ since } \int_G f\,d\lambda = 1,\\ &\leqslant \int_G |f(g)|\left\|\alpha^g(a)-a\right\|_{\mathfrak{A}}\,d\lambda(g)\\ &\leqslant \mathsf{L}(a)\int_G |f(g)|\,d\lambda(g),\end{aligned}$$

as claimed. □

Second, we have the following application of Harmonic analysis.

We note that $\alpha^f\circ\alpha^h = \alpha^{f*g}$.

If G is a compact group, and if π is an irreducible unitary representation of G, then π is finite dimensional. The character ξ of π is the function over G given by $g\in G\mapsto \mathrm{tr}(\pi(a))$, where tr is the normalized trace. We will write $\widehat{G}$ for the set of all characters of G. Now, $\xi*\xi' = 0$ if $\xi,\xi'\in\widehat{G}, \xi\neq\xi'$, and $\xi*\xi=\xi$, so α^ξ is an idempotent on $\mathfrak{A}$.

Definition 1.3.10. The *spectral subspace* of a character $\xi\in\widehat{G}$ of G, for the action α, is the range of α^ξ.

Theorem 1.3.11. *Let ℓ be a continuous function over a compact group G. Let α be a strongly continuous action of G on a unital C*-algebra $\mathfrak{A}$. For all $a\in\mathfrak{sa}(\mathfrak{A})$, we define:*

$$\mathsf{L}(a) = \sup\left\{\frac{\left\|\alpha^g(a)-a\right\|_{\mathfrak{A}}}{\ell(g)} : g\in G, g\neq e\right\},$$

allowing for ∞.

The ordered pair $(\mathfrak{A},\mathsf{L})$ is a quantum compact metric space if, and only if:

$$\left\{a\in\mathfrak{A} : \forall g\in G \quad \alpha^g(a)=a\right\} = \mathbb{C}1_{\mathfrak{A}}.$$

Proof. We already proved the necessary condition. Conversely, assume that the fixed point C*-algebra of α is $\mathbb{C}1_{\mathfrak{A}}$. Denoting $\mathfrak{A}_\chi$ as the range of α^ξ for any character ξ of G. We thus have: $\mathfrak{A} = \mathrm{cl}\left(\oplus_\chi\mathfrak{A}_\chi\right)$. By [24], since the fixed point algebra of α is $\mathbb{C}$, $\mathfrak{A}_\chi$ is finite dimensional for all ξ.

Moreover, if f is a linear combination of characters $\chi_1,\ldots,\chi_n$, of G, then the operator α^f is valued in $\oplus_{j=1}^n\mathfrak{A}_{\chi_j}$. As seen for instance in [54], there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of nonnegative valued linear combinations of characters of G, with norm 1 in $L^1(G,\lambda)$, and such that $\lim_{n\to\infty}\int_G f_n(g)\ell(g)\,d\lambda(g) = \ell(e) = 0$.

Fix $\mu\in\mathscr{S}(\mathfrak{A})$. Let $\varepsilon>0$. There exists $N\in\mathbb{N}$ such that for all $n\geqslant N$, we have $\int_G f_n(g)\ell(g)\,d\lambda(g) < \frac{\varepsilon}{2}$. By Lemma (1.3.9), we conclude that, if $a\in\mathrm{dom}(\mathsf{L})$, $\mathsf{L}(a)\leqslant 1$, then $\left\|a-\alpha^{f_n}(a)\right\|_{\mathfrak{A}} < \frac{\varepsilon}{2}$. On the other hand, the range of α^{f_n} is finite dimensional. Thus $\{a\in\mathrm{dom}(\mathsf{L})\cap\alpha^{f_n}(\mathfrak{A}) : \mathsf{L}(a)\leqslant 1, \mu(a)=0\}$ is totally bounded since it is the ball of some norm in the finite dimensional space $\{a\in\mathfrak{sa}(\mathfrak{A}_n)\}\cap\ker\mu$.

Hence $\{a\in\mathrm{dom}(\mathsf{L}) : \mathsf{L}(a)\leqslant 1, \mu(a)=0\}$ is totally bounded. Since L is lower semicontinuous on $\mathfrak{sa}(\mathfrak{A})$ and $\mathfrak{A}$ is complete, we conclude that $\{a\in\mathrm{dom}(\mathsf{L}) : \mathsf{L}(a)\leqslant 1, \mu(a)=0\}$ is compact. By Theorem (1.1.19), our theorem is proven. □

1.4 LIPSCHITZ MORPHISMS AND QUANTUM ISOMETRIES

Quantum compact metric spaces form a category, when using the following notion of Lipschitz morphisms.

Definition 1.4.1. A k*-Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$, for $k \geqslant 0$, is a unital *-morphism π from $\mathfrak{A}$ to $\mathfrak{B}$ such that

$$\forall b \in \mathfrak{sa}(\mathfrak{B}) \quad \mathsf{L}_{\mathfrak{B}}(\pi(a)) \leqslant k\mathsf{L}_{\mathfrak{A}}(a).$$

Notation 1.4.2. If $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a Lipschitz moprhism between two quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$, then we set:

$$\mathrm{dil}(\pi) := \inf\{k > 0 : \forall a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}}) \quad \mathsf{L}_{\mathfrak{B}} \circ \pi(a) \leqslant k\mathsf{L}_{\mathfrak{A}}(a)\}.$$

Theorem 1.4.3. *Let* $(\mathfrak{A}, \mathsf{L})$ *and* $(\mathfrak{B}, \mathsf{T})$ *be two quantum compact metric spaces. Let* $\pi : \mathfrak{A} \to \mathfrak{B}$ *be a unital *-morphism. The following assertions are equivalent.*

1. π *is a Lipschitz morphism,*
2. $\pi^* : \varphi \in \mathscr{S}(B) \mapsto \varphi \circ \pi \in \mathscr{S}(\mathfrak{A})$ *is a Lipschitz map from* $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{S}})$ *to* $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}})$,
3. $\pi(\mathrm{dom}(\mathsf{L})) \subseteq \mathrm{dom}(\mathsf{S})$.

Moreover, $\mathrm{dil}(\pi) = \mathrm{dil}(\pi^*)$.

Proof. Assume that π is a k-Lipschitz morphism with $k > 0$. Let $\varphi, \psi \in \mathscr{S}(\mathfrak{B})$. Let $a \in \mathrm{dom}(\mathsf{L})$ with $\mathsf{L}(a) \leqslant 1$; therefore, $\mathsf{S}(\pi(a)) \leqslant k\mathsf{L}(a) \leqslant k$. Therefore,

$$\begin{aligned}
|\pi^*(\varphi)(a) - \pi^*(\psi)(a)| &= |\varphi(\pi(a)) - \psi(\pi(a))| \\
&= k\left|\varphi\left(\pi\left(\frac{a}{k}\right)\right) - \psi\left(\pi\left(\frac{a}{k}\right)\right)\right| \\
&= k\,\mathsf{mk}_{\mathsf{S}}(\varphi, \psi).
\end{aligned}$$

Therefore, $\mathsf{mk}_{\mathsf{L}}(\pi^*(\varphi), \pi^*(\psi)) \leqslant k\,\mathsf{mk}_{\mathsf{S}}(\varphi, \psi)$, as claimed.

Now, assume that π^* is k-Lipschitz, with $k > 0$. If $a \in \mathrm{dom}(\mathsf{L})$, with $\mathsf{L}(a) \leqslant 1$, then, using Theorem (1.1.9):

$$\begin{aligned}
\mathsf{S}(\pi(a)) &= \sup\left\{\frac{|\varphi(\pi(a)) - \psi(\pi(a))|}{\mathsf{mk}_{\mathsf{S}}(\varphi, \psi)} : \varphi, \psi \in \mathscr{S}(\mathfrak{B}), \varphi \neq \psi\right\} \\
&= \sup\left\{\frac{|\pi^*(\varphi)(a) - \pi^*(\psi)(a)|}{\mathsf{mk}_{\mathsf{S}}(\varphi, \psi)} : \varphi, \psi \in \mathscr{S}(\mathfrak{B}), \varphi \neq \psi\right\} \\
&\leqslant \sup\left\{\frac{k\mathsf{mk}_{\mathsf{S}}(\varphi, \psi)}{\mathsf{mk}_{\mathsf{S}}(\varphi, \psi)} : \varphi, \psi \in \mathscr{S}(\mathfrak{B}), \varphi \neq \psi\right\} \\
&= k.
\end{aligned}$$

Therefore, $\mathsf{S}(a) \leqslant k\mathsf{L}(a)$ for all $a \in \mathrm{dom}(\mathsf{L})$. If $a \notin \mathrm{dom}(\mathsf{L})$, then $\mathsf{L}(a) = \infty$ and our inequality is trivial. So π is k-Lipschitz.

We have seen that (1) and (2) are equivalent, and indeed $\mathrm{dil}(\pi) = \mathrm{dil}(\pi^*)$ under this assumptions. In turn, this also shows that if either $\mathrm{dil}(\pi) = \infty$ or $\mathrm{dil}(\pi^*) = \infty$, then so is the other. Moreover, we also note that (1) implies (3) trivially.

We conclude by proving that (3) implies (1). Thus, we assume that $\pi(\mathrm{dom}(\mathsf{L})) \subseteq \mathrm{dom}(\mathsf{S})$. For all $a \in \mathfrak{sa}(\mathfrak{A})$, we set $\mathsf{T}(a) = \mathsf{S}(\pi(a))$. Note that $\mathsf{T}(a) < \infty$ for all $a \in \mathrm{dom}(\mathsf{L})$ since $\pi(a) \in \mathrm{dom}(\mathsf{S})$. Since π is continuous, and S is lower semicontinuous on $\mathfrak{sa}(\mathfrak{B})$, we conclude that T is also lower semi-continuous on $\mathfrak{sa}(\mathfrak{A})$. Of course, $\mathsf{T}(1_{\mathfrak{A}}) = 0$ since π is unital.

For all $a \in \mathrm{dom}(\mathsf{L})$, we now set

$$\|a\|_{\mathsf{L}} := \|a\|_{\mathfrak{A}} + \mathsf{L}(a) \text{ and } \|a\|_{\mathsf{T}} := \|a\|_{\mathfrak{A}} + \mathsf{T}(a).$$

By Lemma (1.1.21), both $(\mathrm{dom}(\mathsf{L}), \|\cdot\|_{\mathsf{L}})$ and $(\mathrm{dom}(\mathsf{L}), \|\cdot\|_{\mathsf{T}})$ are Banach spaces. Let $\|\cdot\|_* := \|\cdot\|_{\mathsf{L}} + \|\cdot\|_{\mathsf{T}}$.

If $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{dom}(\mathsf{L})$ converges for the norm $\|\cdot\|_*$, then $(x_n)_{n\in\mathbb{N}}$ is Cauchy for $\|\cdot\|_*$ and therefore, $(x_n)_{n\in\mathbb{N}}$ is Cauchy for both $\|\cdot\|_{\mathsf{L}}$ and $\|\cdot\|_{\mathsf{T}}$. By completeness of $(\mathrm{dom}(\mathsf{L}), \|\cdot\|_{\mathsf{L}})$ and $(\mathrm{dom}(\mathsf{L}), \|\cdot\|_{\mathsf{T}})$, the sequence $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in \mathrm{dom}(\mathsf{L})$ for $\|\cdot\|_{\mathsf{L}}$, and to some $y \in \mathrm{dom}(\mathsf{L})$ for $\|\cdot\|_{\mathsf{T}}$. By construction, since $\|\cdot\|_{\mathfrak{A}} \leqslant \min\{\|\cdot\|_{\mathsf{L}}, \|\cdot\|_{\mathsf{T}}\}$, we then conclude that $(x_n)_{n\in\mathbb{N}}$ converges for $\|\cdot\|_{\mathfrak{A}}$ as well, and thus $x = y$. Therefore, $(x_n)_{n\in\mathbb{N}}$ converges to x for both $\|\cdot\|_{\mathsf{L}}$ and $\|\cdot\|_{\mathsf{T}}$, and therefore, for $\|\cdot\|_*$. So $(\mathrm{dom}(\mathsf{L}), \|\cdot\|_*)$ is a Banach space as well.

Since $\|\cdot\|_{\mathsf{L}} \leqslant \|\cdot\|_*$, the open mapping theorem now implies that there exists $k > 0$ such that $\|\cdot\|_* \leqslant k\|\cdot\|_{\mathsf{L}}$, and thus $\|\cdot\|_{\mathsf{T}} \leqslant k\|\cdot\|_{\mathsf{L}}$.

Let now $a \in \mathrm{dom}(\mathsf{L})$, and let $\mu \in \mathscr{S}(\mathfrak{B})$. We then compute:

$$\begin{aligned}
\mathsf{S}(\pi(a)) &= \mathsf{S}(\pi(a) - \mu(\pi(a))) = \mathsf{S}(\pi(a - \mu(a))) \\
&= \mathsf{T}(a - \mu(a)) = \left\|a - \mu(a)\right\|_{\mathsf{T}} - \left\|a - \mu(a)\right\|_{\mathfrak{A}} \\
&\leqslant k\left\|a - \mu(a)\right\|_{\mathsf{L}} - \left\|a - \mu(a)\right\|_{\mathfrak{A}} \\
&= k\mathsf{L}(a - \mu(a)) + (k-1)\left\|a - \mu(a)\right\|_{\mathfrak{A}} \\
&\leqslant k\mathsf{L}(a) + (k-1)\mathrm{qdiam}(\mathfrak{A}, \mathsf{L})\mathsf{L}(a) \\
&= (k + (k-1)\mathrm{qdiam}(\mathfrak{A}, \mathsf{L}))\mathsf{L}(a).
\end{aligned}$$

Therefore, π is Lipschitz (with $\mathrm{dil}(\pi) \leqslant k + (k-1)\mathrm{qdiam}(\mathfrak{A}, \mathsf{L})$).

Our theorem is thus proven. □

Proposition 1.4.4. *If, for any pair* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ *and* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ *of quantum compact metric spaces, we define* $\mathrm{Hom}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ *the set of all Lipschitz morphisms from* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ *to* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$*, then* $\mathrm{Hom}(\cdot,\cdot)$ *endows the class of quantum compact metric spaces with a category structure, whose morphisms are the Lipschitz morphisms.*

Proof. This is immediate. □

It is immediate to check that an isomorphism between two quantum compact metric spaces $(\mathfrak{A}, \mathsf{L})$ and $(\mathfrak{B}, \mathsf{S})$ in our category of Lipschitz morphisms is given by a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that, $\pi(\mathrm{dom}(\mathsf{L})) = \mathrm{dom}(\mathsf{S})$, and for some constant

$C \geqslant 1$, we have $\frac{1}{C}\mathsf{L} \leqslant \mathsf{S}\circ\pi \leqslant C\mathsf{L}$ on $\mathrm{dom}(\mathsf{L})$. We will however work with a stronger notion of isomorphism in this note.

Quantum isometries are essential examples of Lipschitz morphisms. To motivate the following definition, due to Rieffel [54], we begin by recalling from [50] that if (X,d) is a metric space, if $Z \subseteq X$ is not empty, and if $f : Z \to \mathbb{R}$ is a Lipschitz function over Z, then there exists a Lipschitz function $g : X \to \mathbb{R}$ such that the restriction of g to Z is f, and $\mathrm{dil}(g) = \mathrm{dil}(f)$. Now, let (Y,m) be some other metric space, and let $f : X \to Y$ be an isometry. If $h \in \mathfrak{sa}(C(Y))$ is Lipschitz, then $h\circ f$ is Lipschitz with the same Lipschitz constant; but thanks to the extension result we just recalled, if $h \in \mathfrak{sa}(C(X))$ is Lipschitz, then we can define $h' : f(X) \to \mathbb{R}$ such that $h'(f(x)) = h(x)$ for all $x \in X$ (since f in injective, this is well-defined); of course h' is Lipschitz with $\mathrm{dil}(h') = \mathrm{dil}(h)$ since f is an isometry; we can then extend h' to a function $h'' : Y \to \mathbb{R}$ with $\mathrm{dil}(h'') = \mathrm{dil}(h)$. Now, it is obvious that any function $k : Y \to \mathbb{R}$ such that $k\circ f = h$, we have $\mathrm{dil}(k) \geqslant \mathrm{dil}(h)$ (allowing for $\mathrm{dil}(k) = \infty$). In summary, we note that $\mathrm{dil}(h) = \min\{\mathrm{dil}(k) : k \in \mathfrak{sa}(C(Y)), k\circ f = h\}$. This observation motivates the following definition.

Definition 1.4.5. A *quantum isometry* $\pi : (\mathfrak{A},\mathsf{L}) \to (\mathfrak{B},\mathsf{S})$ is a surjective Lipschitz morphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that

$$\forall b \in \mathrm{dom}(\mathsf{S}) \quad \mathsf{S}(b) = \inf\{\mathsf{L}(a) : \pi(a) = b\}.$$

Note that by Definition (1.4.5), a quantum isometry is automatically a 1-Lipschitz morphism.

Proposition 1.4.6. *If* $\pi : (\mathfrak{A},\mathsf{L}) \to (\mathfrak{B},\mathsf{S})$ *is a quantum isometry between two quantum compact metric spaces* $(\mathfrak{A},\mathsf{L})$ *and* $(\mathfrak{B},\mathsf{S})$*, then* $\pi^* : \varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi\circ\pi \in \mathscr{S}(\mathfrak{A})$ *is an isometry from* $(\mathscr{S}(\mathfrak{B}),\mathsf{mk}_{\mathsf{S}})$ *into* $(\mathscr{S}(\mathfrak{A}),\mathsf{mk}_{\mathsf{L}})$*, and for all* $b \in \mathrm{dom}(\mathsf{S})$*, there exists* $a \in \mathrm{dom}(\mathsf{L})$ *such that* $\mathsf{S}(b) = \mathsf{L}(a)$.

Proof. We first note that:

$$\{b \in \mathrm{dom}(\mathsf{S}) : \mathsf{S}(b) \leqslant 1\} = \pi(\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1\}).$$

Indeed, since π is 1-Lipschitz, we have $\pi(\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1\}) \subseteq \{b \in \mathrm{dom}(\mathsf{S}) : \mathsf{S}(b) \leqslant 1\}$. Let now $b \in \mathrm{dom}(\mathsf{S})$. By Definition (1.4.5), for all $n \in \mathbb{N}$, there exists $a_n \in \mathrm{dom}(\mathsf{L})$ such that $\pi(a_n) = b$ and $\mathsf{L}(a_n) \leqslant \mathsf{S}(b) + \frac{1}{n+1}$. Let $\mu \in \mathscr{S}(\mathfrak{B})$, and note that $\mu\circ\pi \in \mathscr{S}(\mathfrak{A})$. So $\mu\circ\pi(a_n) = \mu(b)$. Since $(\mathfrak{A},\mathsf{L})$ is a quantum compact metric space, the set $C := \{c \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(c) \leqslant \mathsf{S}+1, \mu\circ\pi(c) = 0\}$ is compact in $\mathfrak{sa}(\mathfrak{A})$, so the sequence $(a_n - \mu(b))_{n\in\mathbb{N}}$ has a convergent subsequence with limit in C; so $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence with limit denoted by a. By continuity, $\pi(a) = b$, and by lower semi-continuity, $\mathsf{L}(a) \leqslant \mathsf{S}(b)$. Again since π is a quantum isometry, $\mathsf{S}(b) \leqslant \mathsf{L}(a)$ since $\pi(a) = b$. So $\mathsf{S}(b) = \mathsf{L}(a)$. In turn, this implies $\{b \in \mathrm{dom}(\mathsf{S}) : \mathsf{S}(b) \leqslant 1\} \subseteq \pi(\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1\})$.

Let $\varphi, \psi \in \mathscr{S}(\mathfrak{B})$. We then compute:

$$\begin{aligned}\mathsf{mk}_{\mathsf{L}}(\pi^*(\varphi), \pi^*(\psi)) &= \sup\left\{|\varphi(\pi(a)) - \psi(\pi(a))| : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\right\}\\ &= \sup\left\{|\varphi(b) - \psi(b)| : b \in \mathrm{dom}(\mathsf{S}), \mathsf{S} \leqslant 1\right\}\\ &= \mathsf{mk}_{\mathsf{S}}(\varphi, \psi).\end{aligned}$$

So π^* is an isometry from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{S}})$ to $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}})$. □

Proposition 1.4.7. *Let (X, d_X) and (Y, d_Y) be two compact metric spaces. A function $f : X \to Y$ is an isometry if, and only if, $f^* : h \in C(Y) \mapsto h \circ f \in C(X)$, is a quantum isometry from $(C(X), \mathsf{L}_{d_X})$ to $(C(Y), \mathsf{L}_{d_Y})$.*

Proof. We have shown, prior to Definition (1.4.5), that if $f : X \to Y$ is an isometry, then f^* is a quantum isometry. Assume now that $f : X \to Y$ is a map such that f^* is a quantum isometry. Then $f^{**} : \varphi \in \mathscr{S}(C(X)) \to \mathscr{S}(C(Y))$ is an isometry. For any $x \in X$, write $e_x : f \in C(X) \mapsto f(x)$, and similarly if $y \in Y$. Let $x, y \in X$ — identified with the evaluation maps at x and y, which are states. Then

$$\begin{aligned}d_Y(f(x), f(y)) &= \mathsf{mk}_{\mathsf{L}_{d_Y}}(e_{f(x)}, e_{f(y)}) = \mathsf{mk}_{\mathsf{L}_{d_Y}}(f^{**}(e_x), f^{**}(e_y))\\ &= \mathsf{mk}_{\mathsf{L}_{d_X}}(e_x, e_y) = d_X(x, y).\end{aligned}$$

This concludes our computation. □

The previous proposition does not generalize to the noncommutative setting. First, isometries between state spaces endowed with their respective Monge-Kantorovich metric may not be affine, let alone dual maps to *-morphisms. Second, we note that we encounter a difficulty in trying to establish that even if an isometry is the dual of a *-morphism, things do not quite work in general. Let $(\mathfrak{A}, \mathsf{L})$ and $(\mathfrak{B}, \mathsf{S})$ be two quantum compact metric spaces, and $\pi : \mathfrak{A} \to \mathfrak{B}$ be a unital surjective *-morphism, such that $\pi^* : \mathscr{S}(\mathfrak{B}) \to \mathscr{S}(\mathfrak{A})$ is an isometry from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{S}})$ into $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}})$. Let $b \in \mathrm{dom}(\mathsf{S})$, and $a \in \pi^{-1}(\{b\})$. Using Theorem (1.1.9), we then have:

$$\begin{aligned}\mathsf{S}(b) &= \sup\left\{\frac{|\varphi(b) - \psi(b)|}{\mathsf{mk}_{\mathsf{S}}(\varphi, \psi)} : \varphi, \psi \in \mathscr{S}(\mathfrak{B}), \varphi \neq \psi\right\}\\ &= \sup\left\{\frac{|\varphi(\pi(a)) - \psi(\pi(a))|}{\mathsf{mk}_{\mathsf{S}}(\varphi, \psi)} : \varphi, \psi \in \mathscr{S}(\mathfrak{B}), \varphi \neq \psi\right\}\\ &= \sup\left\{\frac{|\pi^*(\varphi)(a) - \pi^*(\psi)(a)|}{\mathsf{mk}_{\mathsf{S}}(\varphi, \psi)} : \varphi, \psi \in \mathscr{S}(\mathfrak{B}), \varphi \neq \psi\right\}\\ &= \sup\left\{\frac{|\pi^*(\varphi)(a) - \pi^*(\psi)(a)|}{\mathsf{mk}_{\mathsf{L}}(\pi^*(\varphi), \pi^*(\psi))} : \varphi, \psi \in \mathscr{S}(\mathfrak{B}), \varphi \neq \psi\right\}\\ &\leqslant \sup\left\{\frac{|\mu(a) - \nu(a)|}{\mathsf{mk}_{\mathsf{L}}(\mu, \nu)} : \mu, \nu \in \mathscr{S}(\mathfrak{A}), \mu \neq \nu\right\} = \mathsf{L}(a).\end{aligned}$$

So $\inf\{\mathsf{L}(a) : \pi(a) = b\} \geqslant \mathsf{S}(b)$. However, it is not clear how to obtain equality here. Indeed, we can define a function $\hat{a} : \varphi \in \pi^*(\mathscr{S}(\mathfrak{B})) \mapsto \varphi(b)$, and this function is

Lipschitz with Lipschitz constant $\mathsf{S}(b)$, but in order to obtain an element of $\mathrm{dom}(\mathsf{L})$, we would need to find an *affine* extension of $\widehat{a}$ to $\mathscr{S}(\mathfrak{A})$, with the same Lipschitz constant; however McShane's theorem does not provide such an affine extension in general. So, this seems to be the limit of what we can obtain in the noncommutative context. Quantum isometries are a stronger notion than isometries between state spaces, even if we assume them to be dual of *-morphisms.

We can now check that quantum isometries give us a subcategory of the category of quantum compact metric spaces.

Theorem 1.4.8. *If* $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$ *and* $(\mathfrak{D},\mathsf{L}_{\mathfrak{D}})$ *be three quantum compact metric spaces, and if* $\pi:(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})\to(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$ *and* $\varpi:(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})\to(\mathfrak{D},\mathsf{L}_{\mathfrak{D}})$ *are quantum isometries, then* $\varpi\circ\pi$ *is a quantum isometry from* $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ *to* $(\mathfrak{D},\mathsf{L}_{\mathfrak{D}})$.

Proof. Of course, $\varpi\circ\pi$ is a Lipschitz morphism. Now, let $d\in\mathrm{dom}(\mathsf{L}_{\mathfrak{D}})$. If $\varphi(\pi(a))=d$ for some $a\in\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$, then we note that $\mathsf{L}_{\mathfrak{B}}(\pi(a))\geqslant\mathsf{L}_{\mathfrak{D}}(d)$ since ϖ is a quantum isometry, and then $\mathsf{L}_{\mathfrak{A}}(a)\geqslant\mathsf{L}_{\mathfrak{B}}(\pi(a))$ since π is a quantum isometry, so $\mathsf{L}_{\mathfrak{A}}(a)\geqslant\mathsf{L}_{\mathfrak{D}}(d)$.

By Proposition (1.4.6), there exists $b\in\mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ such that $\varpi(b)=d$, and $\mathsf{L}_{\mathfrak{B}}(b)=\mathsf{L}_{\mathfrak{D}}(d)$. Similarly, there exists $a\in\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ such that $\pi(a)=b$ and $\mathsf{L}_{\mathfrak{A}}(a)=\mathsf{L}_{\mathfrak{B}}(b)$. So $\mathsf{L}_{\mathfrak{D}}(d)=\mathsf{L}(a)$ with $\varpi\circ\pi(a)=d$.

Altogether, we have shown that $\mathsf{L}_{\mathfrak{D}}(d)=\min\{\mathsf{L}_{\mathfrak{A}}(a):a\in\mathrm{dom}(\mathsf{L}),\pi(a)=d\}$. This concludes our proof. $\square$

The proper notion of isomorphism between quantum compact metric spaces, for our purpose, is given by isomorphisms in the subcategory of quantum compact metric spaces with quantum isometries as arrows. We are led to the following definition.

Definition 1.4.9. Let $(\mathfrak{A},\mathsf{L})$ and $(\mathfrak{B},\mathsf{S})$ be two quantum compact metric spaces. A *full quantum isometry* $\pi:(\mathfrak{A},\mathsf{L})\to(\mathfrak{B},\mathsf{S})$ from $(\mathfrak{A},\mathsf{L})$ to $(\mathfrak{B},\mathsf{S})$ is a *-isomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ such that $\pi(\mathrm{dom}(\mathsf{L}))=\mathrm{dom}(\mathsf{S})$, and $\mathsf{S}\circ\pi=\mathsf{L}$ on $\mathfrak{sa}(\mathfrak{A})$.

Proposition 1.4.10. *Let* $(\mathfrak{A},\mathsf{L})$ *and* $(\mathfrak{B},\mathsf{S})$ *be two quantum compact metric spaces. A *-morphism* $\pi:\mathfrak{A}\to\mathfrak{B}$ *is a full quantum isometry from* $(\mathfrak{A},\mathsf{L})$ *to* $(\mathfrak{B},\mathsf{S})$ *if, and only if,* π *is a *-isomorphism and a quantum isometry such that* π^{-1} *is also a quantum isometry.*

Proof. If π is a full quantum isometry, then it is a *-isomorphism. Moreover, let $b\in\mathrm{dom}(\mathsf{S})$. By definition, $\mathsf{S}(b)=\mathsf{L}(\pi^{-1}(a))=\inf\{\mathsf{L}(a):\pi(a)=b\}$ since $\pi^{-1}(\{b\})=\{\pi^{-1}(b)\}$. So π is a quantum isometry. The same reasoning applies to π^{-1}, once we note that $\mathsf{S}=\mathsf{L}\circ\pi^{-1}$.

Conversely, assume π is a *-isomorphism such that π and π^{-1} are quantum isometries. It follows that $\pi(\mathrm{dom}(\mathsf{L}))\subseteq\mathrm{dom}(\mathsf{S})$, and $\pi^{-1}(\mathrm{dom}(\mathsf{S}))\subseteq\mathrm{dom}(\mathsf{L})$, so

$$\pi(\mathrm{dom}(\mathsf{L}))\subseteq\mathrm{dom}(\mathsf{S})=\pi(\pi^{-1}(\mathrm{dom}(\mathsf{S})))\subseteq\pi(\mathrm{dom}(\mathsf{L}))$$

so $\pi(\operatorname{dom}(\mathsf{L})) = \operatorname{dom}(\mathsf{S})$. Moreover, let $a \in \operatorname{dom}(\mathsf{L})$. Then $\mathsf{S}(\pi(a)) = \inf\{\mathsf{L}(c) : \pi(c) = a\} = \mathsf{L}(a)$ since π is a quantum isometry, and it is a bijection. This concludes our proof. □

For our purpose, two quantum compact metric spaces are "the same" when there exists a full quantum isometry between them.

1.5 THE LIPSCHITZ DISTANCE

The Lipschitz distance between compact metric spaces [20] provides a distance between homeomorphic compact metric spaces based upon bi-Lipschitz isomorphisms, and thus it is natural to define it in this paper in light of our study of Lipschitz morphisms.

This section provides the noncommutative generalization of the Lipschitz metric, which in essence is a metric on Lip-norms with common domains. The quantum Lipschitz distance is complete and dominates the quantum propinquity when working on appropriate classes of quasi-Leibniz quantum compact metric spaces. The Lipschitz distance also provides natural examples of totally bounded classes for the quantum propinquity, and thus compact classes for the dual propinquity [36].

Notation 1.5.1. Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quantum compact metric spaces and let $\varphi : \mathfrak{A} \to \mathfrak{B}$ a unital *-morphism. We denote by $\mathsf{dil}(\varphi)$ the Lipschitz seminorm of the dual map $\varphi : \mu \in (\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}) \mapsto \mu \circ \varphi \in (\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$, i.e.:

$$\mathsf{dil}(\varphi) = \sup\left\{ \frac{\mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}(\mu \circ \varphi, \nu \circ \varphi)}{\mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}(\mu, \nu)} : \mu, \nu \in \mathscr{S}(\mathfrak{B}), \mu \neq \nu \right\},$$

with the understanding that this quantity may be infinite. We refer to this quantity as the *dilation factor*, or just *dilation* of the given Lipschitz morphism.

Remark 1.5.2. If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are two quantum compact metric spaces with lower semicontinuous Lip-norms, and if $\varphi : \mathfrak{A} \to \mathfrak{B}$ a unital *-morphism, then $\mathsf{dil}(\varphi) = \inf\left\{C > 0 : \mathsf{L}_{\mathfrak{B}} \circ \varphi \leqslant C\mathsf{L}_{\mathfrak{A}}\right\}$ with the usual convention that $\inf \emptyset = \infty$.

Definition 1.5.3. The *Lipschitz distance* between two quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is:

$$\begin{aligned} &\mathsf{LipD}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = \\ &\qquad \inf\left\{\max\left\{\left|\ln(\mathsf{dil}(\varphi))\right|, \left|\ln(\mathsf{dil}(\varphi^{-1}))\right|\right\} : \varphi : \mathfrak{A} \to \mathfrak{B} \text{ is a *-isomorphism}\right\}, \end{aligned}$$

with the conventions that $\inf \emptyset = \infty$ and $\ln(\infty) = \infty$.

Proposition 1.5.4. *If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are two quantum compact metric spaces. Then:*

$$\begin{aligned} &\mathsf{LipD}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = \\ &\qquad \inf\left\{\max\left\{\left|\ln(\mathsf{dil}(\varphi))\right|, \left|\ln(\mathsf{dil}(\varphi^{-1}))\right|\right\} \middle| \begin{array}{l} \varphi : \mathfrak{A} \to \mathfrak{B} \textit{ is a *-isomorphism} \\ \varphi(\operatorname{dom}(\mathsf{L}_{\mathfrak{A}})) = \operatorname{dom}(\mathsf{L}_{\mathfrak{B}}) \end{array} \right\}, \end{aligned}$$

with the convention that $\inf\emptyset = \infty$.

Proof. This follows from our characterization of Lipschitz morphisms. □

The Lipschitz distance between Jordan-Lie quantum compact metric spaces is actually achieved, as established in the following lemma. This observation will prove useful in establishing that the Lipschitz distance in indeed a distance up to quantum isometry.

Lemma 1.5.5. *If* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ *and* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ *are two quantum compact metric spaces such that* $\mathsf{LipD}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) < \infty$ *then there exists a *-isomorphism* $\varphi : \mathfrak{A} \to \mathfrak{B}$ *such that:*

$$\max\left\{|\ln(\mathsf{dil}(\varphi))|, |\ln(\mathsf{dil}(\varphi^{-1}))|\right\} = \mathsf{LipD}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})).$$

Proof. Suppose that $\mathsf{LipD}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = C$ for some $C \geqslant 0$. There exists a sequence of *-isomorphism $(\varphi_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ we have:

$$C^{-1}\exp\left(-\frac{1}{n+1}\right)\mathsf{L}_{\mathfrak{B}}\circ\varphi_n \leqslant \mathsf{L}_{\mathfrak{A}} \leqslant C\exp\left(\frac{1}{n+1}\right)\mathsf{L}_{\mathfrak{B}}\circ\varphi_n.$$

Let $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(a) < \infty$. Since $\|\varphi_n(a)\|_{\mathfrak{B}} = \|a\|_{\mathfrak{A}}$ and $\mathsf{L}_{\mathfrak{B}}\circ\varphi_n(a) \leqslant 2C\mathsf{L}_{\mathfrak{A}}(a)$ for all $n \in \mathbb{N}$, we conclude that $(\varphi_n(a))_{n\in\mathbb{N}}$ admits a convergent subsequence since $\mathsf{L}_{\mathfrak{B}}$ is a Lip-norm. Let $\varphi_\infty(a)$ be its limit; as $\mathsf{L}_{\mathfrak{B}}$ is lower semicontinuous, we conclude that $\mathsf{L}_{\mathfrak{A}}(\varphi_\infty(a)) \leqslant 2C\mathsf{L}_{\mathfrak{A}}(a)$.

Since $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) \leqslant n, \|a\| \leqslant n\}$ is compact for the norm $\|\cdot\|_{\mathfrak{A}}$, hence separable for all $n \in \mathbb{N}$, so is:

$$\mathrm{dom}(\mathsf{L}_{\mathfrak{A}}) = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) < \infty\} = \bigcup_{n\in\mathbb{N}}\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) \leqslant n, \|a\|_{\mathfrak{A}} \leqslant n\}.$$

Let $\mathfrak{F}$ be a countable dense subset of $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) < \infty\}$. A diagonal argument proves that there exists a subsequence $(\varphi_{f(n)})_{n\in\mathbb{N}}$ such that for all $a \in \mathfrak{F}$ we have $(\varphi_{f(n)}(a))_{n\in\mathbb{N}}$ converges uniformly to $\varphi_\infty(a)$ (see [38, Theorem 5.13]).

Moreover, if $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(a) < \infty$, then for all $\varepsilon > 0$, there exists $a_\varepsilon \in \mathfrak{F}$ with $\|a - a_\varepsilon\| < \frac{\varepsilon}{3}$. Let $N \in \mathbb{N}$ be such that for all $p, q \geqslant N$, we have $\|\varphi_{f(p)}(a_\varepsilon) - \varphi_{f(q)}(a_\varepsilon)\|_{\mathfrak{B}} \leqslant \frac{\varepsilon}{3}$. Thus for all $p, q \geqslant N$, we have:

$$\begin{aligned}|\varphi_{f(p)}(a) - \varphi_{f(q)}(a)| &\leqslant |\varphi_{f(p)}(a - a_\varepsilon)| + |\varphi_{f(p)}(a_\varepsilon) - \varphi_{f(q)}(a_\varepsilon)| + |\varphi_{f(q)}(a - a_\varepsilon)| \\ &\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Thus $(\varphi_{f(n)}(a))_{n\in\mathbb{N}}$ converges as well, since it is a Cauchy sequence in $\mathfrak{A}$ which is complete. Its limit is denoted once more by $\varphi_\infty(a)$.

Note that since for all $n \in \mathbb{N}$ and for all $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$, we have $\|\varphi_n(a)\|_{\mathfrak{B}} = \|a\|_{\mathfrak{A}}$, we also have $\|\varphi_\infty(a)\|_{\mathfrak{B}} = \|a\|_{\mathfrak{A}}$. We thus have defined an isometric map $\varphi_\infty : \mathrm{dom}(\mathsf{L}_{\mathfrak{A}}) \to \mathfrak{sa}(\mathfrak{B})$. Moreover, as a pointwise limit of Jordan-Lie morphisms, φ_∞ is also a Jordan-Lie morphisms on $\mathrm{dom}(\mathsf{L})$.

Now $\mathsf{L}_{\mathfrak{B}}$ is lower semi-continuous and, for all $n \in \mathbb{N}$ we have $\mathsf{L}_{\mathfrak{B}}\circ\varphi_n(a) \leqslant C\exp(\frac{1}{n+1})\mathsf{L}_{\mathfrak{A}}(a)$. Thus $\mathsf{L}_{\mathfrak{B}}\circ\varphi_\infty(a) \leqslant C\mathsf{L}_{\mathfrak{A}}(a)$. Thus $\mathsf{dil}(\varphi_\infty) \leqslant C$.

Thus φ_∞ extends by continuity to a Jordan-Lie morphism from $\mathfrak{sa}(\mathfrak{A})$ to $\mathfrak{sa}(\mathfrak{B})$. Our argument is now concluded in the same manner as [38, Claim 5.18, Theorem 5.13] and proves that φ_∞ extends to a unital *-morphism from $\mathfrak{A}$ to $\mathfrak{B}$ with $\mathrm{dil}(\varphi_\infty) \leqslant C$.

The same method may be applied to construct some subsequence of $\left(\varphi_n^{-1}\right)_{n\in\mathbb{N}}$ converging pointwise on $\mathrm{dom}(\mathsf{L}_\mathfrak{B})$ to some *-morphism ψ_∞ on $\mathfrak{B}$ with $\mathsf{L}_\mathfrak{A}\circ\psi_\infty \leqslant C\mathsf{L}_\mathfrak{A}$. Up to extracting further subsequences, we shall henceforth assume that both $(\varphi_{f(n)})_{n\in\mathbb{N}}$ and $(\varphi^{-1}_{f(n)})_{n\in\mathbb{N}}$ converge pointwise to, respectively φ_∞ on $\mathrm{dom}(\mathsf{L}_\mathfrak{A})$ and ψ_∞ on $\mathrm{dom}(\mathsf{L}_\mathfrak{B})$. It is then immediate to check that $\varphi_\infty\circ\psi_\infty$ is the identity of $\mathrm{dom}(\mathsf{L}_\mathfrak{B})$ and $\psi_\infty\circ\varphi_\infty$ is the identity on $\mathrm{dom}(\mathsf{L}_\mathfrak{A})$. Then by construction, $\psi_\infty\circ\varphi_\infty$ is the identity on $\mathfrak{A}$ and $\varphi_\infty\circ\psi_\infty$ is the identity on $\mathfrak{B}$. Thus φ_∞ is a *-isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

In particular, we also obtain that $\mathsf{L}_\mathfrak{A}\circ\varphi_\infty^{-1} \leqslant C\mathsf{L}_\mathfrak{B}$ and thus $\mathsf{dil}(\varphi^{-1}) \leqslant C$. As we may not have both $\mathsf{dil}(\varphi) < C$ and $\mathsf{dil}(\varphi^{-1}) < C$, since C is the infimum of the dilations of such *-isomorphisms, the lemma is proven. □

We now establish that the Lipschitz distance is indeed, a distance up to quantum isometry, and that it dominates the quantum propinquity.

Theorem 1.5.6. *The Lipschitz distance is an extended metric up to quantum isometry on the class of quantum compact metric spaces. Explicitly, for all quantum compact metric spaces* $(\mathfrak{A},\mathsf{L}_\mathfrak{A})$, $(\mathfrak{B},\mathsf{L}_\mathfrak{B})$ *and* $(\mathfrak{D},\mathsf{L}_\mathfrak{D})$, *we have:*

1. $\mathsf{LipD}((\mathfrak{A},\mathsf{L}_\mathfrak{A}),(\mathfrak{B},\mathsf{L}_\mathfrak{B})) \in [0,\infty]$, *and is finite if and only if there exists a *-isomorphism* $\varphi:\mathfrak{A}\to\mathfrak{B}$ *such that* $\varphi(\mathrm{dom}(\mathsf{L}_\mathfrak{A})) = \mathrm{dom}(\mathsf{L}_\mathfrak{B})$,
2. $\mathsf{LipD}((\mathfrak{A},\mathsf{L}_\mathfrak{A}),(\mathfrak{D},\mathsf{L}_\mathfrak{D})) \leqslant \mathsf{LipD}((\mathfrak{A},\mathsf{L}_\mathfrak{A}),(\mathfrak{B},\mathsf{L}_\mathfrak{B})) + \mathsf{LipD}((\mathfrak{B},\mathsf{L}_\mathfrak{B}),(\mathfrak{D},\mathsf{L}_\mathfrak{B}))$,
3. $\mathsf{LipD}((\mathfrak{A},\mathsf{L}_\mathfrak{A}),(\mathfrak{B},\mathsf{L}_\mathfrak{B})) = \mathsf{LipD}((\mathfrak{B},\mathsf{L}_\mathfrak{B}),(\mathfrak{A},\mathsf{L}_\mathfrak{A}))$,
4. $\mathsf{LipD}((\mathfrak{A},\mathsf{L}_\mathfrak{A}),(\mathfrak{B},\mathsf{L}_\mathfrak{B})) = 0$ *if and only if* $(\mathfrak{A},\mathsf{L}_\mathfrak{A})$ *and* $(\mathfrak{B},\mathsf{L}_\mathfrak{B})$ *are fully quantum isometric.*

Proof. The function LipD is valued in $[0,\infty]$ by definition, and finite if and only if there exists a bi-Lipschitz isomorphism between its two arguments. It is symmetric in its two arguments by construction. The triangle inequality follows from simple computations as well.

If there exists an isometric isometry between two compact quantum metric spaces, then their Lipschitz distance is null. Only the converse of this observation requires our assumption that the domain of Lip-norms be Jordan-Lie algebras. We simply apply Lemma (1.5.5). □

Chapter Two

The Gromov-Hausdorff propinquity

2.1 HAUSDORFF DISTANCE

We begin with a notion of how far a point and a nonempty subset of a metric space can be from each other.

Definition 2.1.1. The *distance from $x \in E$ to $A \subseteq E$*, with (E, d) a metric space and $A \neq \emptyset$, is

$$d(x, A) := \inf\left\{d(x, y) : y \in A\right\}.$$

Theorem 2.1.2. *Let (E, d) be a metric space.*

1. $\forall A \subseteq E \quad A \neq \emptyset \implies 0 \leqslant d(x, A)$,
2. $\forall x, y \in E \quad d(x, \{y\}) = d(x, y)$,
3. $\forall A, B \subseteq E \quad A \subseteq B$ *and* $A \neq \emptyset \implies \forall x \in E \quad d(x, B) \leqslant d(x, A)$,
4. $\forall x \in E \quad \forall A \subseteq E \quad d(x, A) = 0 \iff x \in \mathrm{cl}(A)$,
5. $\forall x \in E \quad \forall A \subseteq E \quad d(x, A) = d(x, \mathrm{cl}(A))$,
6. $\forall A \subseteq E \quad A \neq \emptyset \implies \forall x, y \in E \quad \left|d(x, A) - d(y, A)\right| \leqslant d(x, y)$.

Proof. By definition of a distance, 0 is a lower bound of $\{d(x, y) : y \in A\}$ for all $A \subseteq E$ with $A \neq \emptyset$ and $x \in E$. So $0 \leqslant d(x, A)$.

Assertion (2) is obvious.

Let now $A, B \subseteq E$ with $A \subseteq B$. Let $x \in E$. Since $A \subseteq B$, we conclude:

$$\left\{d(x, y) : y \in A\right\} \subseteq \left\{d(x, y) : y \in B\right\}$$

and thus:

$$d(x, A) = \inf\left\{d(x, y) : y \in A\right\} \geqslant \left\{d(x, y) : y \in B\right\} = d(x, B).$$

If $d(x, A) = 0$ for some $x \in X$ then for all $\varepsilon > 0$, there exists $y \in A$ such that $d(x, y) < \varepsilon$, and thus $X(x, \varepsilon) \cap A \neq \emptyset$. As $\varepsilon > 0$ is arbitrary, $x \in \mathrm{cl}(A)$.

Since $A \subseteq \mathrm{cl}(A)$, we have $d(x, \mathrm{cl}(A)) \leqslant d(x, A)$ for all $x \in E$ by Theorem (2.1.2). On the other hand, let $x \in E$ and $\varepsilon > 0$. If $y \in \mathrm{cl}(A)$, then there exists $y_\varepsilon \in A$ such that

$d(y, y_\varepsilon) < \varepsilon$, and thus $d(x, y_\varepsilon) \leqslant d(x, y) + d(y, y_\varepsilon) < d(x, y) + \varepsilon$. So $d(x, A) \leqslant d(x, y) + \varepsilon$ for all $y \in \mathrm{cl}(A)$, i.e. $d(x, A) \leqslant d(x, \mathrm{cl}(A)) + \varepsilon$. As $\varepsilon > 0$ is arbitrary, $d(x, A) \leqslant d(x, \mathrm{cl}(A))$. So $d(x, A) = d(x, \mathrm{cl}(A))$ for all $x \in E$.

Last, let $A \subseteq E$ and let $x, y \in E$. Let now $z \in A$. Since:

$$d(x, A) \leqslant d(x, z) \leqslant d(x, y) + d(y, z)$$

we conclude that since $z \in A$ is arbitrary, $d(x, A) - d(x, y) \leqslant d(y, A)$. Therefore:

$$d(x, A) - d(y, A) \leqslant d(x, y).$$

Similarly, $d(y, A) - d(x, A) \leqslant d(x, y)$, and thus:

$$\left| d(x, A) - d(y, A) \right| \leqslant d(x, y),$$

which completes our proof. □

Lemma 2.1.3. *Let (E, d) be a metric space. If $A, B \subseteq E$ are two nonempty bounded subsets of (E, d), then so is $A \cup B$, and*

$$\forall x \in A \quad d(x, B) \leqslant \mathrm{diam}(A \cup B, d) < \infty.$$

Proof. Fix $x_0 \in A$ and $y_0 \in B$. If $x \in A$ and $y \in B$, then:

$$d(x, y) \leqslant d(x, x_0) + d(x_0, y_0) + d(y_0, y) \leqslant \mathrm{diam}(A, d) + d(x_0, y_0) + \mathrm{diam}(B, d).$$

Of course, if $x, y \in A$, then $d(x, y) \leqslant \mathrm{diam}(A, d)$, and if $x, y \in B$, then $d(x, y) \leqslant \mathrm{diam}(B, d)$. Thus $A \cup B$ is bounded, with diameter at most $d(x_0, y_0) + \mathrm{diam}(A, d) + \mathrm{diam}(B, d)$.

In particular, if $x \in A$, then for all $y \in B$, since $x, y \in A \cup B$, we conclude that $d(x, y) \leqslant \mathrm{diam}(A \cup B, d)$; therefore $d(x, B) \leqslant \mathrm{diam}(A \cup B, d)$.

This concludes our proof. □

In view of Lemma (2.1.3), the following quantity is thus well–defined.

Definition 2.1.4. Let (E, d) be a metric space. The *Hausdorff pseudo-distance* $\mathsf{Haus}[d](A, B)$ between two bounded nonempty subsets $A, B \subseteq E$ of E is defined by:

$$\mathsf{Haus}[d](A, B) = \max\left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\}.$$

The Hausdorff pseudo-distance extends the distance between points.

Lemma 2.1.5. *If (E, d) be a metric space, then*

$$\forall x, y \in E \quad d(x, y) = \mathsf{Haus}[d](\{x\}, \{y\}).$$

Proof. This follows immediately from Definition (2.1.4) and Theorem (2.1.2). □

The closure operator maps bounded subsets to bounded subsets. A key observation is that the extended Hausdorff pseudo-distance does not distinguish between a subset A of a metric space and its closure $\mathrm{cl}(A)$. Now, the Hausdorff pseudo-metric satisfies all the properties of a metric, except that it may be null between two different sets — though, as seen in the following theorem, the Hausdorff distance is zero between two sets *exactly when they have the same closure.*

Theorem 2.1.6. *Let (E,d) be a metric space. The following assertions hold for any bounded nonempty subsets $A,B,C \subseteq E$ of (E,d):*

- $\mathsf{Haus}[d](A,B) = \mathsf{Haus}[d](B,A)$.
- $\mathsf{Haus}[d](A,C) \leqslant \mathsf{Haus}[d](A,B) + \mathsf{Haus}[d](B,C)$.
- $\mathsf{Haus}[d](A,B) = \mathsf{Haus}[d](A,\mathrm{cl}(B)) = \mathsf{Haus}[d](\mathrm{cl}(A),\mathrm{cl}(B))$.
- $\mathsf{Haus}[d](A,B) = 0$ *if and only if* $\mathrm{cl}(A) = \mathrm{cl}(B)$*,*
- $\frac{1}{2}|\mathrm{diam}(A,d) - \mathrm{diam}(B,d)| \leqslant \mathsf{Haus}[d](A,B) \leqslant \mathrm{diam}(A\cup B,d)$.

Proof. By construction, $\mathsf{Haus}[d](A,B) = \mathsf{Haus}[d](B,A)$.

Let $x \in A$. If $\varepsilon > 0$, there exists $y \in Y$ such that $d(x,B) \leqslant d(x,y) + \frac{\varepsilon}{2}$. There exists $z \in C$ such that $d(y,z) \leqslant d(y,C) + \frac{\varepsilon}{2}$. Thus $d(x,z) \leqslant d(x,y) + d(y,z) \leqslant d(x,B) + d(y,C) + \varepsilon$. Consequently:

$$d(x,C) \leqslant d(x,B) + d(y,C) \leqslant \mathsf{Haus}[d](A,B) + \mathsf{Haus}[d](B,C) + \varepsilon.$$

Thus $\sup_{x\in A} d(x,C) \leqslant \mathsf{Haus}[d](A,B) + \mathsf{Haus}[d](B,C) + \varepsilon$. By symmetry, we also have $\sup_{z\in C} d(z,A) \leqslant \mathsf{Haus}[d](A,B) + \mathsf{Haus}[d](B,C) + \varepsilon$ and thus, as claimed:

$$\mathsf{Haus}[d](A,C) \leqslant \mathsf{Haus}[d](A,B) + \mathsf{Haus}[d](B,C) + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we conclude

$$\mathsf{Haus}[d](A,C) \leqslant \mathsf{Haus}[d](A,B) + \mathsf{Haus}[d](B,C),$$

as desired.

Let now $A,B \subseteq X$. By Theorem (2.1.2), we have $d(\cdot,A) = d(\cdot,\mathrm{cl}(A))$, so $\sup_{x\in B} d(x,A) = \sup_{x\in B} d(x,\mathrm{cl}(A))$. On the other hand, let $x \in \mathrm{cl}(A)$. For all $\varepsilon > 0$, there exists $x_\varepsilon \in A$ such that $d(x,x_\varepsilon) < \varepsilon$. Thus, for all $y \in B$, we have $d(x,y) \leqslant d(x,x_\varepsilon) + d(x_\varepsilon,y) < \varepsilon + \mathsf{Haus}[d](A,B)$. So $\sup_{x\in\mathrm{cl}(A)} d(x,B) \leqslant \mathsf{Haus}[d](A,B) + \varepsilon$. As $\varepsilon > 0$ is arbitrary, we conclude that $\sup_{x\in\mathrm{cl}(A)} d(x,B) \leqslant \mathsf{Haus}[d](A,B)$. Therefore, $\mathsf{Haus}[d](A,B) = \mathsf{Haus}[d](\mathrm{cl}(A),B)$.

We then compute:

$$\begin{aligned}\mathsf{Haus}[d](A,B) &= \mathsf{Haus}[d](\mathrm{cl}(A),B) = \mathsf{Haus}[d](B,\mathrm{cl}(A))\\ &= \mathsf{Haus}[d](\mathrm{cl}(B),\mathrm{cl}(A)) = \mathsf{Haus}[d](\mathrm{cl}(A),\mathrm{cl}(B)).\end{aligned}$$

In particular, $\mathsf{Haus}[d](A,B) = 0$ if and only if $\mathsf{Haus}[d]\mathrm{cl}(A),\mathrm{cl}(B)) = 0$.

If $\mathsf{Haus}[d](\mathrm{cl}(A), \mathrm{cl}(B)) = 0$ then, for all $x \in \mathrm{cl}(A)$, we have $d(x, \mathrm{cl}(B)) = 0$, so $x \in \mathrm{cl}(\mathrm{cl}(B)) = \mathrm{cl}(B)$ and thus $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$, and by symmetry, $\mathrm{cl}(B) \subseteq \mathrm{cl}(A)$ so $\mathrm{cl}(A) = \mathrm{cl}(B)$. Of course, if $\mathrm{cl}(A) = \mathrm{cl}(B)$ then $\mathsf{Haus}[d](\mathrm{cl}(A), \mathrm{cl}(B)) = 0$.

By Lemma (2.1.3), we have shown that $\mathsf{Haus}[d](A, B) \leqslant \mathrm{diam}(A \cup B, d)$. On the other hand, by Theorem (2.1.2), we also have, for all $x, x' \in A$ and $y, y' \in B$, that:

$$d(x, x') \leqslant d(x, y) + d(y, y') + d(y', x') \leqslant 2\mathsf{Haus}[d](A, B) + \mathrm{diam}(B, d),$$

and therefore $\mathrm{diam}(A, d) - \mathrm{diam}(B, d) \leqslant 2\mathsf{Haus}[d](A, B)$. Switching the role of A and B, we obtain the desired inequalities. □

We therefore conclude that the Hausdorff pseudo-distance is indeed a distance on the set of all bounded subset of a metric space (E, d), which are equal to their closures.

Corollary 2.1.7. *If (E, d) is a metric space, the restriction of $\mathsf{Haus}[d]$ to the set*

$$\mathrm{Hyper}(E, d) = \{A \subseteq E : A \neq \emptyset, \mathrm{cl}(A) = A \textit{ and } A \textit{ is bounded}\}$$

of bounded closed nonempty subsets *of (E, d) is a distance, called the* Hausdorff distance. *The space* $\mathrm{Hyper}(E, d)$ *is called the* hyperspace *of (E, d).*

Proof. This follows immediately from Theorem (2.1.6). □

Notation 2.1.8. If (E, d) is a metric space, and if $A \subseteq E$ with $A \neq \emptyset$, then $\mathrm{diam}(A, d) := \sup\left\{d(x, y) : x, y \in A\right\}$.

Theorem 2.1.9. *If (E, d) is a metric space, then the function $\mathrm{diam}(\cdot, d)$ is continuous on $(\mathrm{Hyper}(E, d), \mathsf{Haus}[d])$.*

Proof. Fix $A \in \mathrm{Hyper}(E, d)$. Let $\varepsilon > 0$. If $B \in \mathrm{Hyper}(E, d)$ and $\mathsf{Haus}[d](A, B) < \frac{\varepsilon}{2}$, then for all $x, y \in A$ (resp. B), there exists $z, t \in B$ (resp. A) such that $d(x, z) < \frac{\varepsilon}{2}$ and $d(y, t) < \frac{\varepsilon}{2}$. Thus $d(x, y) \leqslant d(z, t) + \varepsilon \leqslant \mathrm{diam}(B, d) + \varepsilon$ (resp. $\mathrm{diam}(A, d) + \varepsilon$). Thus $|\mathrm{diam}(A, d) - \mathrm{diam}(B, d)| < \varepsilon$. This concludes our proof. □

We now check that various metric properties are shared between a space and its hyperspace.

Theorem 2.1.10. *Let (E, d) be a metric space. (E, d) is complete if, and only if*

$$(\mathrm{Hyper}(E, d), \mathsf{Haus}[d])$$

is complete.

Proof. Assume (E, d) is complete. Let $(A_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $(\mathrm{Hyper}(E, d), \mathsf{Haus}[d])$. Up to extracting a subsequence, we assume that $\mathsf{Haus}[d](\mathfrak{A}_n, \mathfrak{A}_{n+1}) < \frac{1}{2^n}$. Let now

$$A := \left\{\lim_{n \to \infty} x_n : (x_n)_{n \in \mathbb{N}} \text{ converges }, \forall n \in \mathbb{N} \quad x_n \in A_n\right\}.$$

First, A is not empty. Pick $x_0 \in A_0$. Assume that, for some $n \in \mathbb{N}$, we have constructed $x_k \in A_k$ with $d(x_k, x_{k+1}) < \frac{1}{2^k}$ for all $k \in \{0, \dots, n\}$. Since $\mathsf{Haus}[d](A_n, A_{n+1}) < frac12^n$, there exists $x_{n+1} \in A_{n+1}$ such that $d(x_n, x_{n+1}) < \frac{1}{2^n}$. Hence $(x_n)_{n\in\mathbb{N}}$ is Cauchy, and therefore, since (E, d) is complete, it converges. So $A \neq \emptyset$.

Now, let $\varepsilon > 0$. Let $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have $\frac{1}{2^{n-1}} < \varepsilon$. Let $x \in A$. There exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in A_n$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$. Thus there exists $M \in \mathbb{N}$ such that, for all $n \geqslant M$, we have $d(x, x_n) < \frac{\varepsilon}{2}$. If $M \leqslant N$, then for all $n \geqslant N$, we have $d(x_n, x) < \varepsilon$. If $N < M$, then for all $n \geqslant N$, there exists $y \in A_n$ such that $d(y, x_M) < \frac{1}{2^n} < \frac{\varepsilon}{2}$, and thus $d(x, y) \leqslant d(x, x_M) + d(x_M, y) < \varepsilon$. Either way, for all $x \in A$ and for all $n \geqslant N$, there exists $y \in A_n$ such that $d(x, y) < \varepsilon$.

On the other hand, fix $n \geqslant N$. Let $x \in A_n$. Proceeding as above, we construct a sequence $(x_k)_{k\geqslant n}$ with $x_n = x$, $d(x_k, x_{k+1}) < \frac{1}{2^k}$ for all $k \geqslant n$, and $x_k \in A_k$ for all $k \in \mathbb{N}$. Thus $(x_k)_{k\in\mathbb{N}}$ is Cauchy — hence convergence, since (E, d) is complete. Let $l = \lim_{k\to\infty} x_k$, so $l \in A$. We have

$$d(l, x) = \lim_{k\to\infty} d(x_k, x) \leqslant \limsup_{k\to\infty} \sum_{j=n+1}^{k} d(x_j, x_{j-1}) = \frac{1}{2^{n-1}} < \varepsilon.$$

So, for all $n \geqslant N$, we have shown that $\mathsf{Haus}[d](A, A_n) < \varepsilon$, and thus $(A_n)_{n\in\mathbb{N}}$ converges to A for $\mathsf{Haus}[d]$.

Let now assume that $(\mathrm{Hyper}(E, d), \mathsf{Haus}[d])$ is complete. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (E, d). Then $(\{x_n\})_{n\in\mathbb{N}}$ is Cauchy, hence convergent, for $\mathsf{Haus}[d]$. Now, since $\mathrm{diam}(\{x_n\}, d) = 0$ for all $n \in \mathbb{N}$, we also conclude the limit of $(\{x_n\})_{n\in\mathbb{N}}$ is also a singleton $\{y\}$. It is now easy to check that y is the limit of $(x_n)_{n\in\mathbb{N}}$. □

Theorem 2.1.11. *Let (E, d) be a metric space. (E, d) is totally bounded space if, and only if, $(\mathrm{Hyper}(E, d), \mathsf{Haus}[d])$ is totally bounded.*

Proof. Assume that (E, d) is totally bounded. Let $\varepsilon > 0$. Let G be a finite $\frac{\varepsilon}{2}$-dense subset of (E, d). Let F be the set obtained as all possible unions of the closed balls centered at points in G, so G is finite. Let now $A \subseteq E$ be not empty, bounded and closed. For each $x \in A$, let $t(x) \in F$ be chosen so that $d(x, t(x)) < \frac{\varepsilon}{2}$. By construction, the set $B := \bigcup_{x\in A} E[t(x), \frac{\varepsilon}{2}]$ is an element of G. If $x \in A$, then there exists $t(x) \in B$ such that $d(x, t(x)) < \frac{\varepsilon}{2} < \varepsilon$. Let now $y \in B$. By construction, there exists $x \in A$ such that $y \in E[t(x), \frac{\varepsilon}{2}]$. So $d(y, x) \leqslant d(y, t(x)) + d(t(x), x) < \varepsilon$.

Hence, $\mathsf{Haus}[d](A, B) < \varepsilon$. As G is finite, $(\mathrm{Hyper}(E, d), \mathsf{Haus}[d])$ is totally bounded.

The map $x \in E \mapsto \{x\}$ is an isometric embedding, so if $(\mathrm{Hyper}(E, d), \mathsf{Haus}[d])$ is totally bounded, so is (E, d). □

Corollary 2.1.12. *Let (E, d) be a metric space. (E, d) is a compact metric space if, and only if $(\mathrm{Hyper}(E, d), \mathsf{Haus}[d])$ is a compact metric space.*

Proof. (E, d) is compact if, and only if it is totally bounded and complete, if and only if $(\mathrm{Hyper}(E, d), \mathsf{Haus}[d])$ is totally bounded and complete. □

Last, we check that the Hausdorff distance on $\mathrm{Hyper}(E,d)$ induces a topology which actually only depend on the topology, rather than the metric, of (E,d), when (E,d) is compact.

Theorem 2.1.13. *Let X be a compact space with topology τ and let $\mathscr{F} = \{U^c : U \in \tau, U \neq X\}$ be the set of all nonempty closed subsets of X. The* Vietoris topology *is the smallest topology on $\mathscr{F}$ generated by the topological basis*

$$\mathscr{O}(U, V_1, \ldots, V_n) = \left\{F \in \mathscr{F} : F \subseteq U \text{ and } \forall j \in \{1, \ldots, n\} \quad F \cap V_j \neq \emptyset\right\}$$

for all $n \in \mathbb{N}$ and $U, V_1, \ldots, V_n \in \tau$.

If d is a metric on X which induced τ, then the topology induced by $\mathsf{Haus}[\mathrm{d}]$ on $\mathrm{Hyper}(X,\mathrm{d})$ is the Vietoris topology.

Consequently, if d_1 and d_2 are two metrics which induce the same topology on X then $\mathsf{Haus}[\mathrm{d}_1]$ and $\mathsf{Haus}[\mathrm{d}_2]$ induce the same topology on $\mathscr{F}$.

Proof. Let $F \in \mathscr{F}$ and $r > 0$. Since F is compact, there exists $x_1, \ldots, x_n \in F$ for some $n \in \mathbb{N}$ such that $F \subseteq \bigcup_{j=1}^n X\left(x_j, \frac{r}{2}\right)$ where the open ball in (X,d) of center any $y \in X$ and radius r is denoted by $X(y,r)$. For all $j \in \{1, \ldots, n\}$, we set $V_j = X\left(x_j, \frac{r}{2}\right)$.

Let $U = \bigcup_{j=1}^n V_j$. Note that by construction, $F \in \mathscr{O}(U, V_1, \ldots, V_n)$. Now let $G \in \mathscr{O}(U, V_1, \ldots, V_n)$. If $x \in G$, then $x \in U$ and thus $x \in V_j$ for some $j \in \{1, \ldots, n\}$, implying that $\mathrm{d}(x,F) < \frac{r}{2}$. If $x \in F$, then $x \in V_j$ for some $j \in \{1, \ldots, n\}$. Since $G \cap V_j \neq \emptyset$, there exists $y \in G \cap V_j$ and by definition of V_j, we conclude $\mathrm{d}(x,y) < r$. Hence $\mathsf{Haus}[\mathrm{d}](F,G) < r$. Thus $\mathscr{O}(U, V_1, \ldots, V_n) \subseteq \mathscr{F}(F,r)$.

Let now $U, V_1, \ldots, V_n \in \tau$ be given with $F \in \mathscr{O}(U, V_1, \ldots, V_n)$. Since $X \setminus U$ is closed and disjoint from F, we conclude that there exists $\varepsilon_0 > 0$ such that, for all $x \in F$ and $y \in X \setminus U$, we have $\mathrm{d}(x,y) \geqslant \varepsilon_0$.

Now, for each $j \in \{1, \ldots, n\}$, there exists $x_j \in F \cap V_j$ and there exists $\varepsilon_j > 0$ such that $X(x_j, \varepsilon_j) \subseteq V_j$. Let $\varepsilon = \min\{\varepsilon_j : j \in \{0, \ldots, n\}\}$.

Let $G \in \mathscr{F}(F,\varepsilon)$. Let $x \in G$. There exists $y \in F$ such that $\mathrm{d}(x,y) < \varepsilon$. Thus $x \in U$ since $\mathrm{d}(x,y) < \varepsilon_0$. Thus $G \subseteq U$.

Let $j \in \{1, \ldots, n\}$. There exists $y \in G$ such that $\mathrm{d}(x_j,y) < \varepsilon \leqslant \varepsilon_j$, and thus by construction, $y \in X(x_j, \varepsilon_j) \subseteq V_j$ and thus $G \cap V_j \neq \emptyset$. We thus have shown that $G \in \mathscr{O}(U, V_1, \ldots, V_n)$. Thus $\mathscr{F}(F,\varepsilon) \subseteq \mathscr{O}(U, V_1, \ldots, V_n)$.

This proves our lemma. $\square$

The Hausdorff distance is defined with reference to a fixed, based metric space. If we want to talk about convergence of different metric spaces, which are not subspaces of a fixed metric space — an issue which will become all the more relevant in the noncommutative geometry setting — we then turn to an idea due to Edwards [15] (for compact metric spaces) and Gromov [20, 19] (for locally compact metric spaces). We focus our attention to the compact case.

$$\begin{tikzcd}
 & (Z, d_Z) & \\
(X, d_X) \arrow[ur, hook, "\iota_X"] & & (Y, d_Y) \arrow[ul, hook', "\iota_Y"']
\end{tikzcd}$$

FIGURE 2.1: Gromov-Hausdorff Isometric Embeddings

$$\begin{tikzcd}
 & (\mathscr{S}(\mathfrak{D}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}) & \\
 & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \arrow[u, dashed, two heads] \arrow[ddl, two heads, "\pi_{\mathfrak{A}}"'] \arrow[ddr, two heads, "\pi_{\mathfrak{B}}"] & \\
(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}) \arrow[uur, hook, bend left, "\pi^*_{\mathfrak{A}}"] & & (\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}) \arrow[uul, hook', bend right, "\pi^*_{\mathfrak{B}}"'] \\
(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \arrow[u, dashed, two heads] & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \arrow[u, dashed, two heads]
\end{tikzcd}$$

FIGURE 2.2: A tunnel and the dual isometric embeddings of state spaces

$\hookrightarrow$	isometry
$\twoheadrightarrow$	quantum isometry
dotted arrows	duality relations
$\pi^* : \varphi \mapsto \varphi \circ \pi$	dual map
$(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$, $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$	K-quantum compact metric spaces

Definition 2.1.14. If (X, d_X) and (Y, d_Y) are two compact metric spaces, then we define the *Gromov-Hausdorff distance* between (X, d_X) and (Y, d_Y) as:

$$\begin{aligned}\mathsf{GH}((X, d_X), (Y, d_Y)) := \inf\Big\{ & \mathsf{Haus}[d_Z](i_X(X), i_Y(Y)) : (Z, d_X) \text{ compact metric space },\\ & i_X : X \to Z \text{ and } i_Y : Y \to Z \text{ isometries} \Big\}.\end{aligned}$$

We will construct a noncommutative analogue of this metric.

2.2 TUNNELS AND BRIDGES

Tunnels generalize the idea of an isometric embedding of two quantum compact metric spaces into a third one to our noncommutative metric geometric setting. They form the basis for the construction of the propinquity.

Definition 2.2.1. Let $(\mathfrak{A}_1, \mathsf{L}_1)$ and $(\mathfrak{A}_2, \mathsf{L}_2)$ be two K-quantum compact metric spaces. A *K-tunnel* $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_1, \pi_2)$ from $(\mathfrak{A}_1, \mathsf{L}_1)$ to $(\mathfrak{A}_2, \mathsf{L}_2)$ is a given by a K-quantum compact metric space $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$, and two quantum isometries $\pi_1 : (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \twoheadrightarrow (\mathfrak{A}_1, \mathsf{L}_1)$ and $\pi_2 : (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \twoheadrightarrow (\mathfrak{A}_2, \mathsf{L}_2)$.

The *domain* $\mathrm{dom}(\tau)$ of τ is $(\mathfrak{A}_1, \mathsf{L}_1)$, and the co-domain $\mathrm{codom}(\tau)$ of τ is $(\mathfrak{A}_2, \mathsf{L}_2)$. The *inverse* of τ is $\tau^{-1} := (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_2, \pi_1)$.

Notation 2.2.2. If $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_1, \pi_2)$ is a tunnel from $(\mathfrak{A}_1, \mathsf{L}_1)$ to $(\mathfrak{A}_2, \mathsf{L}_2)$, we will sometimes write:

$$\tau : (\mathfrak{A}_1, \mathsf{L}_1) \xleftarrow{\pi_1} (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \xrightarrow{\pi_2} (\mathfrak{A}_2, \mathsf{L}_2).$$

With this notation,

$$\tau^{-1} : (\mathfrak{A}_2, \mathsf{L}_2) \xleftarrow{\pi_2} (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \xrightarrow{\pi_1} (\mathfrak{A}_1, \mathsf{L}_1).$$

We can associate a number to any tunnel, which will measure how far apart its domain and co-domain are from its perspective.

Definition 2.2.3. The *extent* $\chi(\tau)$ of a tunnel $\tau : (\mathfrak{A}_1, \mathsf{L}_1) \xleftarrow{\pi_1} (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \xrightarrow{\pi_2} (\mathfrak{A}_2, \mathsf{L}_2)$ between two quantum compact metric spaces $(\mathfrak{A}_1, \mathsf{L}_1)$ and $(\mathfrak{A}_2, \mathsf{L}_2)$ is defined as the nonnegative number:

$$\max_{j \in \{1,2\}} \mathsf{Haus}\left[\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}\right]\left(\mathscr{S}(\mathfrak{D}), \left\{\varphi \circ \pi_j : \varphi \in \mathscr{S}(\mathfrak{A}_j)\right\}\right),$$

where $\mathsf{Haus}\left[\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}\right]$ is the Hausdorff distance by the Monge-Kantorovich metric $\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}$ on the class of the weak* closed subsets of the state space $\mathscr{S}(\mathfrak{D})$ of $\mathfrak{D}$.

Of course, for any tunnel, $\chi\left(\tau^{-1}\right) = \chi(\tau)$.

Our first task is to show that tunnels always exist between any two quantum compact metric spaces. To this end, we introduce the concept of a bridge: a bridge is in a sense dual to a tunnel. There are several notions of bridges, and we choose to focus on a special kind which, as we shall see later, can be used to directly compute an upper bound for the extent of a tunnel.

Definition 2.2.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two quantum compact metric spaces. A *bridge* $\gamma := (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ from $\mathfrak{A}$ to $\mathfrak{B}$ is given by a unital C*-algebra $\mathfrak{D}$, two unital *-morphisms $\pi_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{D}$ and $\pi_{\mathfrak{B}} : \mathfrak{B} \to \mathfrak{D}$, and a self-adjoint element $\omega \in \mathfrak{sa}(\mathfrak{D})$ such that there exists a state $\varphi \in \mathscr{S}(\mathfrak{D})$ with $\varphi(\omega \cdot) = \varphi(\cdot \omega) = \varphi$.

The element ω is called the *pivot* of the bridge γ. The *bridge seminorm* $\mathsf{bn}_{\gamma}(\cdot, \cdot)$ associated with γ is defined by $\mathsf{bn}_{\gamma}(a, b) = \|\pi_{\mathfrak{A}}(a)\omega - \omega \pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}}$, for all $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$.

Remark 2.2.5. In [38], bridges were introduced where the *-morphisms were required to also be injective. This is not necessary here, or even in [38], as the height, defined below, will avoid degenerate situations by being “large” for bridges with non-injective *-morphisms.

A first quantity associated with bridges, called the reach, enables us to define a tunnel from a bridge.

Definition 2.2.6. Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quantum compact metric spaces. The *reach* $\mathsf{reach}\left(\gamma \middle| \mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right)$ of a bridge $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is

$$\begin{aligned}\mathsf{reach}\left(\gamma \middle| \mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right) := \mathsf{Haus}\left[\mathfrak{D}\right]\Big(&\{a\omega : a \in \mathrm{dom}\left(\mathsf{L}_{\mathfrak{A}}\right), \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1\}, \\ &\{\omega b : b \in \mathrm{dom}\left(\mathsf{L}_{\mathfrak{B}}\right), \mathsf{L}_{\mathfrak{B}}(b) \leqslant 1\}\Big).\end{aligned}$$

Of course, if we associate a tunnel to a bridge, it is natural to ask what the extent of that tunnel would be, based on the bridge used to construct it. To this end, we introduce the following quantity.

Definition 2.2.7. Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quantum compact metric spaces. The *height* $\mathsf{reach}\left(\gamma \middle| \mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right)$ of a bridge $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is

$$\mathsf{height}\left(\gamma \middle| \mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right) := \max\{\mathsf{Haus}\left[\mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}\right](\mathscr{S}(\mathfrak{A})), \{\varphi \circ \pi_{\mathfrak{A}} : \varphi \in \mathscr{S}(\mathfrak{D}|\omega\}),$$

where $\mathscr{S}(\mathfrak{D}|\omega) := \{\varphi \in \mathscr{S}(\mathfrak{A}) : \varphi(\omega \cdot) = \varphi(\cdot \omega) = \varphi\}$.

Theorem 2.2.8. *Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two K-quantum compact metric spaces. If $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is a bridge from $\mathfrak{A}$ to $\mathfrak{B}$, then by setting $\mathfrak{T} := \mathfrak{A} \oplus \mathfrak{B}$, letting $\varepsilon \geqslant \mathsf{reach}\left(\gamma \middle| \mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right)$ and, for all $(s, b) \in \mathfrak{sa}(\mathfrak{T})$:*

$$\mathsf{T}(a, b) := \max\left\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b), \frac{1}{\varepsilon}\mathsf{bn}_{\gamma}(a, b)\right\},$$

and $\rho_{\mathfrak{A}}(a, b) := a$, $\rho_{\mathfrak{B}}(a, b) := b$, the quadruple $(\mathfrak{T}, \mathsf{T}, \rho_{\mathfrak{A}}, \rho_{\mathfrak{B}})$ is a K-tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$, whose extent is at most

$$\lambda\left(\gamma \middle| \mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right) := \mathsf{reach}\left(\gamma \middle| \mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right) + \mathsf{height}\left(\gamma \middle| \mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right).$$

Proof. The function T_{ε} is easily checked to be a seminorm, and, as the maximum of two lower semi-continuous functions (L_1 and L_2) and one continuous function over $\mathfrak{sa}(\mathfrak{A})$ ($\frac{1}{\varepsilon}\mathsf{bn}_{\gamma}(\cdot)$), it is lower semi-continuous over $\mathfrak{sa}(\mathfrak{F})$.

We also note that $\mathrm{dom}(\mathsf{T}_{\varepsilon}) = \mathrm{dom}(\mathsf{L}_{\mathfrak{A}}) \oplus \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$, which is dense in $\mathfrak{sa}(\mathfrak{F})$ since $\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ is dense in $\mathfrak{sa}(\mathfrak{A})$ and $\mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ is dense in $\mathfrak{sa}(\mathfrak{B})$.

Now, let $(a, b) \in \mathfrak{sa}(\mathfrak{F})$. If $\mathsf{T}_{\varepsilon}(a, b) = 0$, then $\mathsf{L}_{\mathfrak{A}}(a) = 0$ so $a \in \mathbb{R}1_{\mathfrak{A}}$, and $\mathsf{L}_{\mathfrak{B}}(b) = 0$ so $b \in \mathbb{R}1_{\mathfrak{B}}$; moreover $\pi_{\mathfrak{A}}(a) = \pi_{\mathfrak{B}}(b)$. Therefore, $(a, b) \in \mathbb{R}(1_{\mathfrak{A}}, 1_{\mathfrak{B}}) = \mathbb{R}1_{\mathfrak{F}}$.

We now establish the Leibniz property.

If $a_1, a_2 \in \mathfrak{sa}(\mathfrak{A})$ and $b_1, b_2 \in \mathfrak{sa}(\mathfrak{B})$, then:

$$\begin{aligned}\frac{1}{\varepsilon}\mathsf{bn}_{\gamma}(a_1 a_2, b_1 b_2) &= \frac{1}{\varepsilon}\|\pi_{\mathfrak{A}}(a_1 a_2)\omega - \omega\pi_{\mathfrak{B}}(b_1 b_2)\|_{\mathfrak{D}} \\ &\leqslant \frac{1}{\varepsilon}\|\pi_{\mathfrak{A}}(a_1)\pi_{\mathfrak{A}}(a_2)\omega - \pi_{\mathfrak{A}}(a_1)\omega\pi_{\mathfrak{B}}(b_2)\|_{\mathfrak{D}} \\ &\quad + \frac{1}{\varepsilon}\|\pi_{\mathfrak{A}}(a_1)\omega\pi_{\mathfrak{B}}(b_2) - \omega\pi_{\mathfrak{B}}(b_2)\pi_{\mathfrak{B}}(b_2)\|_{\mathfrak{D}} \\ &\leqslant \frac{\|a_1\|_{\mathfrak{A}}}{\varepsilon}\|\pi_{\mathfrak{A}}(a_2)\omega - \omega\pi_{\mathfrak{B}}(b_2)\|_{\mathfrak{D}} + \frac{\|b_2\|_{\mathfrak{B}}}{\varepsilon}\|\pi_{\mathfrak{A}}(a_1)\omega - \omega\pi_{\mathfrak{B}}(a_1)\|_{\mathfrak{D}} \\ &\leqslant \|(a_1, b_1)\|_{\mathfrak{F}}\frac{\mathsf{bn}_{\gamma}(a_2, b_2)}{\varepsilon} + \|(a_2, b_2)\|_{\mathfrak{F}}\frac{\mathsf{bn}_{\gamma}(a_1, b_1)}{\varepsilon},\end{aligned} \tag{2.2.1}$$

since $\|a,b\|_{\mathfrak{F}} = \max\{\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{B}}\}$ for all $(a,b) \in \mathfrak{F}$. As $\mathsf{bn}_\gamma(\cdot)$ is a seminorm on $\mathfrak{A}\oplus\mathfrak{B}$, it follows from Inequality (2.2.1) that

$$\begin{aligned}\mathsf{bn}_\gamma(\Re(a_1a_2, b_1b_2)) &\leqslant \|(a_1,b_1)\|_{\mathfrak{F}}\,\mathsf{bn}_\gamma(()a_2,b_2) + \|(a_2,b_2)\|_{\mathfrak{F}}\,\mathsf{bn}_\gamma(()a_1,b_1)\\ &\leqslant \varepsilon\left(\|(d_1,d_2)\|_{\mathfrak{F}}\,\mathsf{T}_\varepsilon(e_1,e_2) + \mathsf{T}_\varepsilon(d_1,d_2)\,\|(e_1,e_2)\|_{\mathfrak{F}}\right).\end{aligned}$$

Since $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ is a K-quantum compact metric space, we also have:

$$\begin{aligned}\mathsf{L}_{\mathfrak{A}}(\Re(a_1a_2)) &\leqslant K(\|a_1\|_{\mathfrak{A}}\,\mathsf{L}_{\mathfrak{A}}(a_2) + \mathsf{L}_{\mathfrak{A}}(a_1)\,\|a_2\|_{\mathfrak{A}} + \mathsf{L}_{\mathfrak{A}}(a_1)\mathsf{L}_{\mathfrak{A}}(a_2))\\ &\leqslant K(\|(a_1,b_1)\|_{\mathfrak{F}}\,\mathsf{T}_\varepsilon(a_2,b_2) + \mathsf{T}_\varepsilon(a_1,b_1)\,\|(a_2,b_2)\|_{\mathfrak{F}} + \mathsf{T}_\varepsilon(a_1,b_1)\mathsf{T}_\varepsilon(a_2,b_2)),\end{aligned}$$

and similarly with $\mathsf{L}_{\mathfrak{B}}(\Re(b_1b_2))$, so overall, since $K\geqslant 1$,we obtain:

$$\mathsf{T}_\varepsilon(\Re(a_1a_2,b_1b_2)) \leqslant K(\|(a_1,b_1)\|_{\mathfrak{F}}\,\mathsf{T}_\varepsilon(a_2,b_2)+\mathsf{T}_\varepsilon(a_1,b_1)\,\|(a_2,b_2)\|_{\mathfrak{F}}+\mathsf{T}_\varepsilon(a_1,b_1)\mathsf{T}_\varepsilon(a_2,b_2)).$$

A similar computation applies with $\Re$ replaced by $\Im$.

Let $\mu \in \mathscr{S}(\mathfrak{A})$, extended to $\mathfrak{F}$ by setting $\mu(a,b) := \mu(a)$ for all $(a,b)\in\mathfrak{F}$. Let $(a,b) \in \mathfrak{sa}(\mathfrak{F})$ with $\mathsf{T}_\varepsilon(a,b) \leqslant 1$ and $\mu(a,b) = 0$. By Lemma (1.1.14), since $\mathsf{L}_{\mathfrak{A}}(a) \leqslant 1$ and $\mu(a) = 0$, we have $\|a\|_{\mathfrak{A}} \leqslant \mathrm{qdiam}(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$. Moreover $\|b\|_{\mathfrak{B}} \leqslant \varepsilon + \|b\|_{\mathfrak{B}}$ by definition of T_ε. So

$$\begin{aligned}&\left\{(a,b)\in\mathfrak{sa}(\mathfrak{F}) : \mathsf{T}_\varepsilon(a,b)\leqslant 1, \mu(a,b)=0\right\}\subseteq\\ &\{x\in\mathrm{dom}(\mathsf{L}_{\mathfrak{A}}) : \mathsf{L}_{\mathfrak{A}}(x)\leqslant 1, \mu(x)=0\}\times\{y\in\mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) : \mathsf{L}(y)\leqslant 1, \left\|y\right\|_{\mathfrak{B}}\leqslant \mathrm{qdiam}(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})+\varepsilon\}.\end{aligned}$$

Since $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ is a quantum compact metric space, by Theorem (1.1.19), the set $\{x\in\mathrm{dom}(\mathsf{L}_{\mathfrak{A}}) : \mathsf{L}_{\mathfrak{A}}(x)\leqslant 1, \mu(x)=0\}$ is compact. By Theorem (1.1.22), since $(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$ is a quantum compact metric space, the set $\{y \in \mathrm{dom}(\mathsf{L}_2) : \mathsf{L}(y) + \left\|y\right\|_{\mathfrak{B}} \leqslant 1 + \mathrm{qdiam}(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})+\varepsilon\}$ is also compact. So $\left\{(a,b)\in\mathfrak{sa}(\mathfrak{F}) : \mathsf{T}_\varepsilon(a,b)\leqslant 1, \mu(a,b)=0\right\}$ is totally bounded; as it is closed and $\mathfrak{F}$ is complete, it is in fact compact.

By Theorem (1.1.19), we conclude that $(\mathfrak{F},\mathsf{T}_\varepsilon)$ is a K-quantum compact metric space.

Let now $a\in\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ with $\mathsf{L}_{\mathfrak{A}}(a) = 1$. By definition of the bridge reach of γ, there exists $b\in\mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ with $\mathsf{L}_{\mathfrak{B}}(b)\leqslant 1$ and $\mathsf{bn}_\gamma(a,b)\leqslant\varepsilon$. By construction, we then see that $\mathsf{T}_\varepsilon(a,b) = 1 = \mathsf{L}_{\mathfrak{A}}(a))$. Therefore, $p_{\mathfrak{A}}$ is a quantum isometry (note that if $\mathsf{L}_{\mathfrak{A}}(a) = 0$, then $a = t1_{\mathfrak{A}}$ for some t and then we can choose $b = t1_{\mathfrak{B}}$). A similar computation proves that $p_{\mathfrak{B}}$ is also a quantum isometry. So $(\mathfrak{F},\mathsf{T}_\varepsilon,p_{\mathfrak{A}},p_{\mathfrak{B}})$ is a K-tunnel from $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$.

So far, only the reach of our bridge γ played any role. It is also possible, using the bridge height, to find an upper bound on the extent of τ. Let $\varphi\in\mathscr{S}(\mathfrak{F})$. Therefore, there exists $t\in[0,1]$, $\mu\in\mathscr{S}(\mathfrak{A})$ and $\nu\in\mathscr{S}(\mathfrak{B})$ such that $\varphi = t\mu\circ p_{\mathfrak{A}} + (1-t)\nu\circ p_{\mathfrak{B}}$. Now, by definition of the height of γ, there exists $\theta\in\mathscr{S}(\mathfrak{D}|\omega)$ such that $\theta\circ\pi_{\mathfrak{B}}\in\mathscr{S}(\mathfrak{B})$ and $\mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}(\nu,\theta\circ\pi_{\mathfrak{B}}) \leqslant \mathsf{height}\left(\gamma\middle|\mathsf{L}_{\mathfrak{A}},\mathsf{L}_{\mathfrak{B}}\right)$. Now, if $(a,b)\in\mathfrak{F}$ with $\mathsf{T}(a,b)\leqslant 1$, then in particular, $\mathsf{L}_{\mathfrak{B}}(b) \leqslant 1$ and $\|\pi_{\mathfrak{A}}(a)\omega - \omega\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}} \leqslant \mathsf{reach}\left(\gamma\middle|\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right)$. Therefore:

$$\begin{aligned}
|\varphi(a,b) - (t\mu + (1-t)\theta\circ\pi_{\mathfrak{A}})\circ p_{\mathfrak{A}}(a,b)| &\leqslant |t\mu(a) + (1-t)\nu(b) - t\mu(a) - (1-t)\theta\circ\pi_{\mathfrak{A}}(a)| \\
&\leqslant |\nu(b) - \theta(\pi_{\mathfrak{A}}(a))| \\
&\leqslant |\nu(b) - \theta(\pi_{\mathfrak{B}}(b))| + |\theta(\pi_{\mathfrak{B}}(b)) - \theta(\pi_A(a))| \\
&= |\nu(b) - \theta(\pi_{\mathfrak{B}}(b))| + |\theta(\omega\pi_{\mathfrak{B}}(b)) - \theta(\pi_A(a)\omega)| \\
&\leqslant \mathsf{height}\left(\gamma\middle|\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right) + \mathsf{bn}_{\gamma}(a,b) \\
&\leqslant \mathsf{height}\left(\gamma\middle|\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right) + \mathsf{reach}\left(\gamma\middle|\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right).
\end{aligned}$$

So $\mathsf{mk}_{\mathsf{T}}(\varphi, (t\mu + (1-t)\theta\circ\pi_{\mathfrak{A}})) \leqslant \lambda\left(\gamma\middle|\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right)$, noting that $t\mu + (1-t)\theta\circ\pi_{\mathfrak{A}}$ is a state of $\mathfrak{A}$.

A symmetric reasoning shows that there exists a state ψ of $\mathfrak{B}$ such that $\mathsf{mk}_{\mathsf{T}}(\varphi, \psi\circ p_{\mathfrak{B}}) \leqslant \lambda\left(\gamma\middle|\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right)$. In conclusion, $\chi(\tau) \leqslant \lambda\left(\gamma\middle|\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right)$, as claimed. □

Remark 2.2.9. Theorem (2.2.8) is very helpful for constructing tunnels; to this end, only the reach of the bridge is involved. Sometimes, it may be easier to compute the extent of the resulting tunnel directly, rather than involve the height.

Remark 2.2.10. Let $F : [0,\infty)^4 \to [0,\infty)$ be an increasing function for the product order on $[0,\infty)^4$, with the additional property that $F(x,y,l_x,l_y) \geqslant xl_y + yl_x + l_xl_y$ for all $x,y,l_x,l_y \geqslant 0$. Say that a tunnel $\tau := (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_1, \pi_2)$ is an F-tunnel when $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$ is F-Leibniz, in the sense of Remark (1.1.26). Then it is easy to check that if $(\mathfrak{A}_1, \mathsf{L}_1)$ and $(\mathfrak{A}_2, \mathsf{L}_2)$ are F-quantum compact metric spaces, and γ is a bridge from $\mathfrak{A}$ to $\mathfrak{B}$, then the tunnel $\tau(\gamma)$ from Theorem (2.2.8) is an F-tunnel.

We now apply Theorem (2.2.8) to prove that there always exists a tunnel between any two quantum compact metric spaces.

Lemma 2.2.11. *If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are two quantum compact metric spaces, then there exists a tunnel τ from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$, with*

$$\chi(\tau) \leqslant \max\left\{\mathrm{qdiam}(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), \mathrm{qdiam}(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.$$

Proof. Let $C := \max\{\mathrm{qdiam}(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), \mathrm{qdiam}(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\}$.

Since $\mathfrak{A}$ and $\mathfrak{B}$ are separable C*-algebras, they both admit a unital faithful *-representation on an infinite dimensional countable Hilbert space; up to unitary equivalence, we can thus assume that there exist two unital *-representations $\pi_{\mathfrak{A}}$ and $\pi_{\mathfrak{B}}$ of, respectively, $\mathfrak{A}$ and $\mathfrak{B}$, on $\ell^2(\mathbb{N})$. By definition, $\gamma := (\mathfrak{B}(\ell^2(\mathbb{N})), 1, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is a bridge from $\mathfrak{A}$ to $\mathfrak{B}$. Now, fix $\mu \in \mathscr{S}(\mathfrak{A})$. if $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ with $\mathsf{L}_{\mathfrak{A}}(a) \leqslant 1$, then $\left|\left|\left|\pi_{\mathfrak{A}}(a) - \pi_{\mathfrak{B}}(\mu(a))\right|\right|\right|_{\ell^2(\mathbb{N})} = \left\|a - \mu(a)\right\|_{\mathfrak{A}} \leqslant \mathrm{qdiam}(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \leqslant C$. Of course, $\mathsf{L}_{\mathfrak{B}}(\mu(a)) = 0 \leqslant 1$. Similarly, if we fix $\nu \in \mathscr{S}(\mathfrak{B})$, we get $\mathsf{bn}_{\gamma}(\nu(b), b) \leqslant C$. So $\mathsf{reach}\left(\gamma\middle|\mathsf{L}_{\mathfrak{A}}, \mathsf{L}_{\mathfrak{B}}\right) \leqslant C$.

From Theorem (2.2.8), we thus deduce that if we set:

$$\mathfrak{D} := \mathfrak{A} \oplus \mathfrak{B} \text{ and } \mathsf{T} : (a,b) \in \mathfrak{D} \mapsto \max\left\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{B}}(b), \frac{1}{C}|||\pi_{\mathfrak{A}}(a) - \pi_{\mathfrak{B}}(b)|||_{\ell^2(\mathbb{N})}\right\},$$

and if $p_{\mathfrak{A}} : (a,b) \in \mathfrak{D} \mapsto a$ and $p_{\mathfrak{B}} : (a,b) \in \mathfrak{D} \mapsto b$, then $\tau := (\mathfrak{D}, \mathsf{T}, p_{\mathfrak{A}}, p_{\mathfrak{B}})$ is a tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. It remains to compute a bound for its extent, which can be done if we compute the height of γ. Since $\pi_{\mathfrak{A}}$ is faithful, and since $\mathscr{S}(\mathfrak{B}(\ell^2(\mathbb{N}))|1) = \mathscr{S}(\mathfrak{B}(\ell^2(\mathbb{N})))$, we conclude that $\pi_{\mathfrak{A}}^*(\mathscr{S}(\mathfrak{B}(\ell^2(\mathbb{N}))|1)) = \mathscr{S}(\mathfrak{A})$, and similarly with $\mathfrak{B}$ in place of $\mathfrak{A}$. So $\mathrm{height}\left(\gamma\middle|\mathsf{L}_{\mathfrak{A}},\mathsf{L}_{\mathfrak{B}}\right) = 0$. So the extent of τ is at most $\lambda\left(\gamma\middle|\mathsf{L}_{\mathfrak{A}},\mathsf{L}_{\mathfrak{B}}\right) \leqslant C$. This concludes our proof. □

The advantage of tunnels over bridges is that they behave like generalized morphisms, with the departure from being given by actual morphisms is measured by the extent. Bridges can not be composed naturally, unlike tunnels. So, in general, bridges are very useful to construct tunnels and obtain bounds on their extent, but tunnels are essential to obtain a good construction of the propinquity as a complete metric, as we shall see. Tunnels, of course, need not be constructed from bridges, and are thus more general and flexible when needed.

2.3 TUNNELS AS SET-VALUED GENERALIZED MORPHISMS

Tunnels behave, informally, as some form of “almost morphisms”. As a first such property, we prove that we can compose tunnels up to $\varepsilon > 0$, in the following sense.

Theorem 2.3.1. *Let $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$, $(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$ and $(\mathfrak{E},\mathsf{L}_{\mathfrak{E}})$ be three K-quantum compact metric spaces for some $K \geqslant 1$. Let $\tau_1 : (\mathfrak{A},\mathsf{L}_{\mathfrak{A}}) \xleftarrow{\pi_1} (\mathfrak{D}_1,\mathsf{L}_1) \xrightarrow{\pi_2} (\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$ be a K-tunnel from $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$, and let $\tau_2 : (\mathfrak{B},\mathsf{L}_{\mathfrak{B}}) \xleftarrow{\rho_1} (\mathfrak{D}_2,\mathsf{L}_2) \xrightarrow{\rho_2} (\mathfrak{E},\mathsf{L}_{\mathfrak{E}})$ be a K-tunnel from $(\mathfrak{B},\mathsf{L}_{\mathfrak{B}})$ to $(\mathfrak{E},\mathsf{L}_{\mathfrak{E}})$. Let $\varepsilon > 0$.*

If we set $\mathfrak{F} := \mathfrak{D}_1 \oplus \mathfrak{D}_2$, and for all $(d_1,d_2) \in \mathfrak{sa}(\mathfrak{F})$, we set:

$$\mathsf{T}_\varepsilon(d_1,d_2) = \max\left\{\mathsf{L}_1(d_1),\mathsf{L}_2(d_2),\frac{1}{\varepsilon}\left\|\pi_2(d_1)-\rho_1(d_2)\right\|_{\mathfrak{B}}\right\},$$

and if $\eta_1 : (d_1,d_2) \in \mathfrak{F} \mapsto d_1 \in \mathfrak{D}_1$ and $\eta_2 : (d_1,d_2) \in \mathfrak{F} \mapsto d_2 \in \mathfrak{D}_2$, then $(\mathfrak{F},\mathsf{T}_\varepsilon)$ is a K-quantum compact metric space, and

$$\tau_1 \circ_\varepsilon \tau_2 : (\mathfrak{A},\mathsf{L}_{\mathfrak{A}}) \xleftarrow{\pi_1\circ\eta_1} (\mathfrak{F},\mathsf{T}_\varepsilon) \xrightarrow{\rho_2\circ\eta_2} (\mathfrak{E},\mathsf{L}_{\mathfrak{E}})$$

is a K-tunnel from $(\mathfrak{A},\mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{E},\mathsf{L}_{\mathfrak{E}})$, whose extent satisfies:

$$\chi(\tau_1 \circ_\varepsilon \tau_2) \leqslant \chi(\tau_1) + \chi(\tau_2) + \varepsilon.$$

Moreover, the affine maps $\varphi \in \mathscr{S}(\mathfrak{D}_1) \mapsto \varphi\circ\eta_1$ and $\varphi \in \mathscr{S}(\mathfrak{D}_2) \mapsto \varphi\circ\eta_2$ are isometries from, respectively, $(\mathscr{S}(\mathfrak{D}_1),\mathsf{mk}_{\mathsf{L}_1})$ and $(\mathscr{S}(\mathfrak{D}_2),\mathsf{mk}_{\mathsf{L}_2})$ into $(\mathscr{S}(\mathfrak{D}_1\oplus\mathfrak{D}_2),\mathsf{mk}_{\mathsf{L}})$.

Proof. Let $\gamma = (\mathfrak{B},1_{\mathfrak{B}},\pi_2,\rho_1)$. By assumption, γ is a bridge from $\mathfrak{D}_1$ to $\mathfrak{D}_2$. Since π_2 and ρ_1 are both quantum isometries, we conclude that

$$\begin{aligned}\{\pi_2(d) : d \in \mathrm{dom}(\mathsf{L}_1), \mathsf{L}_1(d) \leqslant 1\} &= \{b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) : \mathsf{L}_{\mathfrak{B}}(b) \leqslant 1\}\\ &= \{\rho_1(d) : d \in \mathrm{dom}(\mathsf{L}_2), \mathsf{L}_2(d) \leqslant 1\}\end{aligned}$$

so $\mathrm{reach}\left(\gamma\middle|\mathsf{L}_{\mathfrak{A}},\mathsf{L}_{\mathfrak{B}}\right) = 0$. Hence, for all $\varepsilon > 0$, by Theorem (2.2.8), we conclude that $(\mathfrak{F},\mathsf{T}_\varepsilon)$ is a quantum compact metric space, and moreover, η_1 and η_2 are both quantum isometries from $(\mathfrak{F},\mathsf{T}_\varepsilon)$ onto, respectively, $(\mathfrak{D}_1,\mathsf{L}_1)$ and $(\mathfrak{D}_2,\mathsf{L}_2)$.

By Theorem (1.4.8), the *-morphisms $\pi_1 \circ \eta_1$ and $\rho_2 \circ \eta_2$ are quantum isometries from $(\mathfrak{F}, \mathsf{T}_\varepsilon)$ to, respectively, $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{E}, \mathsf{L}_{\mathfrak{E}})$. Therefore, $\tau_1 \circ_\varepsilon \tau_2$, as defined, is indeed a K-tunnel.

It remains to compute the extent of $\tau_1 \circ_\varepsilon \tau_2$. We proceed directly. Let $\varphi \in \mathscr{S}(\mathfrak{F})$. Since $\mathfrak{F} = \mathfrak{D}_1 \oplus \mathfrak{D}_2$, there exists $t \in [0,1]$, $\varphi_1 \in \mathscr{S}(\mathfrak{D}_1)$ and $\varphi_2 \in \mathscr{S}(\mathfrak{D}_2)$ such that $\varphi = t\varphi_1 \circ \eta_1 + (1-t)\varphi_2 \circ \eta_2$.

By definition of the extent of τ_1, there exists $\psi \in \mathscr{S}(\mathfrak{B})$ such that:

$$\mathsf{mk}_{\mathsf{L}}(\varphi_1 \circ \eta_1, \psi \circ \pi_2 \circ \eta_1) = \mathsf{mk}_{\mathsf{L}_1}(\varphi_1, \psi \circ \pi_2) \leqslant \chi(\tau_1).$$

Now, by definition of the extent of τ_2, there exists $\theta \in \mathscr{S}(\mathfrak{E})$ such that:

$$\mathsf{mk}_{\mathsf{L}}(\psi \circ \rho_1 \circ \eta_2, \theta \circ \rho_2 \circ \eta_2) = \mathsf{mk}_{\mathsf{L}_2}(\psi \circ \rho_1, \theta \circ \rho_2) \leqslant \chi(\tau_2).$$

Let $(d_1, d_2) \in \mathfrak{F}$ with $\mathsf{L}(d_1, d_2) \leqslant 1$. Then by construction of L, we have:

$$\mathsf{L}_1(d_1) \leqslant 1, \mathsf{L}_2(d_2) \leqslant 1 \text{ and } \left\|\pi_2(d_1) - \rho_1(d_2)\right\|_{\mathfrak{B}} \leqslant \varepsilon.$$

Thus:

$$\begin{aligned}
|\varphi_1(\eta_1(d_1,d_2)) - \theta(\rho_2 \circ \eta_2(d_1,d_2))| \\
&= |\varphi_1(d_1) - \theta(\pi_2(d_2))| \\
&\leqslant |\varphi_1(d_1) - \psi(\pi_2(d_1))| + |\psi(\pi_2(d_1)) - \psi(\rho_1(d_2))| \\
&\quad + |\psi(\rho_1(d_2)) - \theta(\rho_2(d_2))| \\
&\leqslant \mathsf{mk}_{\mathsf{L}_1}(\varphi_1, \psi \circ \pi_2) + \varepsilon + \mathsf{mk}_{\mathsf{L}_2}(\psi \circ \rho_1, \theta \circ \rho_2) \\
&\leqslant \chi(\tau_1) + \varepsilon + \chi(\tau_2).
\end{aligned}$$

Thus:

$$\mathsf{mk}_{\mathsf{L}}(\varphi_1 \circ \eta_1, \theta \circ \rho_2 \circ \eta_2) \leqslant \chi(\tau_1) + \chi(\tau_2) + \varepsilon.$$

By definition of the extent of τ_2, we can find $\theta_2 \in \mathscr{S}(\mathfrak{E})$ such that:

$$\mathsf{mk}_{\mathsf{L}}(\varphi_2 \circ \eta_2, \theta_2 \circ \rho_2 \circ \eta_2) = \mathsf{mk}_{\mathsf{L}_2}(\varphi_2, \theta_2 \circ \rho_2) \leqslant \chi(\tau_2).$$

Since the Monge-Kantorovich metric mk_{L} is convex in each of its variable by construction, we conclude:

$$\begin{aligned}
\mathsf{mk}_{\mathsf{L}}(\varphi, (t\theta + (1-t)\theta_2) \circ \rho_2 \circ \eta_2) &\leqslant \max\left\{\chi(\tau_1) + \varepsilon + \chi(\tau_2), \chi(\tau_2)\right\} \\
&= \chi(\tau_1) + \varepsilon + \chi(\tau_2),
\end{aligned}$$

and we note that $t\theta + (1-t)\theta_2 \in \mathscr{S}(\mathfrak{E})$. Thus, as $\varphi \in \mathscr{S}(\mathfrak{F})$ was arbitrary, we conclude:

$$\mathsf{Haus}\left[\mathsf{mk}_{\mathsf{L}}\right](\mathscr{S}(\mathfrak{F}), (\rho_2 \circ \eta_2)^*(\mathscr{S}(\mathfrak{E}))) \leqslant \chi(\tau_1) + \chi(\tau_2) + \varepsilon.$$

By symmetry, we would obtain in the same manner that for any $\varphi \in \mathscr{S}(\mathfrak{F})$ there exists $\theta \in \mathscr{S}(\mathfrak{A})$ with:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \theta \circ \pi_1 \circ \eta_1) \leqslant \chi(\tau_1) + \chi(\tau_2) + \varepsilon.$$

Therefore, by Definition (2.2.3):

$$\chi(\tau_1 \circ_\varepsilon \tau_2) \leqslant \chi(\tau_1) + \chi(\tau_2) + \varepsilon,$$

which concludes our theorem. □

Remark 2.3.2. If τ_1 and τ_2 are F-tunnels in the sense of Remark (2.2.10), then Remark (2.2.10) implies that $\tau_1 \circ_\varepsilon \tau_2$ is also an F-tunnel.

Tunnels can in fact be used to define compact-set valued maps between quantum compact metric spaces. The properties of these set valued functions are reminiscent of morphisms, with some nebulosity controlled by the extent of the tunnel.

Definition 2.3.3. Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quantum compact metric spaces, and let $\tau : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xleftarrow{\pi_{\mathfrak{A}}} (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \xrightarrow{\pi_{\mathfrak{B}}} (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be a tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

For all $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$, and for all $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, the *target set* $\mathfrak{t}_\tau(a|l)$ is defined as the subset of $\mathfrak{B}$

$$\mathfrak{t}_\tau(a|l) := \{\pi_{\mathfrak{B}}(d) : d \in \mathrm{dom}(\mathsf{L}_{\mathfrak{D}}), \mathsf{L}(d) \leqslant l, \pi_{\mathfrak{A}}(d) = a\}.$$

We first observe that target sets are not empty.

Lemma 2.3.4. *Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quantum compact metric spaces, and let $\tau : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xleftarrow{\pi_{\mathfrak{A}}} (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \xrightarrow{\pi_{\mathfrak{B}}} (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be a tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.*

For any $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(a) < \infty$ and $r \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, the set $\mathfrak{t}_\tau(a|r)$ is not empty. Moreover, if $l \geqslant r \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, then $\mathfrak{t}_\tau(a|r) \subseteq \mathfrak{t}_\tau(a|l)$.

Proof. Fix $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$.

By definition, if $l \geqslant r$, then $\mathfrak{t}_\tau(a|r) \subseteq \mathfrak{t}_\tau(a|l)$. By Proposition (1.4.6), since $\pi_{\mathfrak{A}}$ is a quantum isometry, there exists $d \in \mathrm{dom}(\mathsf{L}_{\mathfrak{D}})$ such that $\pi_{\mathfrak{A}}(d) = a$ and $\mathsf{L}_{\mathfrak{D}}(d) = a$. Since $\pi_{\mathfrak{B}}$ is also a quantum isometry, we conclude that $\mathsf{L}_{\mathfrak{B}}(\pi_{\mathfrak{B}}(d)) \leqslant \mathsf{L}_{\mathfrak{D}}(d) = \mathsf{L}_{\mathfrak{A}}(a)$. Thus $\pi_{\mathfrak{B}}(d) \in \mathfrak{t}_\tau(a|\mathsf{L}(a))$. This proves target sets are not empty. □

We now prove what is, in a sense, the key property of target sets: a form of continuity, controlled by the extent of a tunnel, from which many other properties of target sets depend. The crucial property of the target sets for a tunnel τ is that their diameters are controlled by the length τ. Consequently, when two Leibniz quantum compact metric spaces are close for the dual Gromov-Hausdorff propinquity, then one may expect that target sets for appropriately chosen tunnels have diameters of the order of the distance between our two quantum compact metric spaces. Thus, if two quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are in fact at distance zero, one may find a sequence of target sets for any $a \in \mathfrak{sa}(\mathfrak{A})$ associated to tunnels of ever smaller length, which converges to a singleton: the element in this singleton would then be our candidate as an image for a by some prospective full quantum iometry. This general intuition will be the base for our proof of Theorem (2.4.3). We now state the fundamental property of target sets upon which all our estimates rely.

Theorem 2.3.5. *Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quantum compact metric spaces, and let $\tau : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xleftarrow{\pi_{\mathfrak{A}}} (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \xrightarrow{\pi_{\mathfrak{B}}} (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be a tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. Let $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$, and let $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$.*

If $d \in \pi_{\mathfrak{A}}^{-1}(\{a\}) \cap \mathfrak{sa}(\mathfrak{D})$ and $\mathsf{L}_{\mathfrak{D}}(a) \leqslant l$, then

$$\|d\|_{\mathfrak{D}} \leqslant \|a\|_{\mathfrak{A}} + l\chi(\tau).$$

Therefore, for all $b \in \mathsf{t}_\tau(a|l)$,

$$\|b\|_{\mathfrak{B}} \leqslant \|a\|_{\mathfrak{A}} + l\chi(\tau) \text{ and } \mathsf{L}_{\mathfrak{B}}(b) \leqslant l.$$

Proof. Let $d \in \pi_{\mathfrak{A}}^{-1}(\{a\}) \cap \mathfrak{sa}(\mathfrak{D})$ with $\mathsf{L}_{\mathfrak{D}}(a) \leqslant l$. Let $\varphi \in \mathscr{S}(\mathfrak{A})$. By Definition (2.2.3), there exists $\psi \in \mathscr{S}(\mathfrak{A})$ such that $\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}(\varphi, \psi\circ\pi_{\mathfrak{A}}) < \chi(\tau)$. Therefore,

$$\begin{aligned} |\varphi(d)| &\leqslant |\varphi(d) - \psi\circ\pi_{\mathfrak{A}}(d)| + |\psi\circ\pi_{\mathfrak{A}}(d)| \\ &\leqslant l\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}(\varphi, \psi\circ\pi_{\mathfrak{A}}) + |\psi(a)| \\ &\leqslant l\chi(\tau) + \|a\|_{\mathfrak{A}}. \end{aligned}$$

Now, if $b \in \mathsf{t}_\tau(a|l)$, then $b = \pi_{\mathfrak{B}}(d)$ for some $d \in \pi_{\mathfrak{A}}^{-1}(\{a\}) \cap \mathfrak{sa}(\mathfrak{D})$ with $\mathsf{L}_{\mathfrak{D}}(a) \leqslant l$; therefore, $\|b\|_{\mathfrak{B}} \leqslant \|d\|_{\mathfrak{B}} \leqslant l\chi(\tau) + \|a\|_{\mathfrak{A}}$ and, since $\pi_{\mathfrak{B}}$ is a quantum isometry, $\mathsf{L}_{\mathfrak{B}}(b) \leqslant \mathsf{L}_{\mathfrak{D}}(d) \leqslant l$. Our proof is complete. □

As a first application, we see that target sets are compact, which is a form of "point-like" behavior.

Proposition 2.3.6. *Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quantum compact metric spaces, and let $\tau : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xleftarrow{\pi_{\mathfrak{A}}} (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \xrightarrow{\pi_{\mathfrak{B}}} (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be a tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. If $a \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$, and if $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, then $\mathsf{t}_\tau(a|l)$ is compact in $\mathfrak{B}$.*

Proof. We prove a little more. By Theorem (2.3.5), we note that, if $\mathfrak{s} := \{d \in \pi_{\mathfrak{A}}^{-1}(\{a\}) : \mathsf{L}_{\mathfrak{D}}(d) \leqslant l\}$, then $\mathfrak{s} \subseteq \{d \in \operatorname{dom}(\mathsf{L}_{\mathfrak{D}}) : \mathsf{L}_{\mathfrak{D}}(d) + \|d\|_{\mathfrak{D}} \leqslant 1 + \|a\|_{\mathfrak{A}} + l\chi(\tau)\}$. The latter set is compact by Theorem (1.1.22). Therefore, $\mathfrak{s}$ is totally bounded. On the other hand, $\pi_{\mathfrak{A}}^{-1}(\{a\})$ is closed since $\pi_{\mathfrak{A}}$ is continuous and $\{a\}$ is closed, and $\{d \in \operatorname{dom}(\mathsf{L}_{\mathfrak{D}}) : \mathsf{L}_{\mathfrak{D}}(d) \leqslant l\}$ is closed since $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$ is a quantum compact metric space. Therefore, σ is closed. Since $\mathfrak{D}$ is complete, $\mathfrak{s}$ is compact.

By definition, $\mathsf{t}_\tau(a|l) = \pi_{\mathfrak{B}}(\mathfrak{s})$ and $\pi_{\mathfrak{B}}$ is continuous, so $\mathsf{t}_\tau(a|l)$ is compact in $\mathfrak{B}$, as claimed. □

Remark 2.3.7. Using the notations of Definition (2.3.3), we note that if $a \in \mathfrak{sa}(\mathfrak{A})$ is not in the domain of $\mathsf{L}_{\mathfrak{A}}$, i.e. $\mathsf{L}_{\mathfrak{A}}(a) = \infty$ with our convention, then $\mathsf{t}_\tau(a|\infty)$ is not empty since $\pi_{\mathfrak{A}}$ is surjective, though it is not compact in general.

We can now relate the linear structure of quantum compact metric spaces and target sets of tunnels. An important consequence of this relation regards the diameter of target sets.

Theorem 2.3.8. *Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quantum compact metric spaces and let $\tau : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xleftarrow{\pi_{\mathfrak{A}}} (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \xrightarrow{\pi_{\mathfrak{B}}} (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be a tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.*

Let $a, a' \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$. If $r \geqslant \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{A}}(a')\}$, then:

1. *For all $b \in \mathsf{t}_\tau(a|r)$, $b' \in \mathsf{t}_\tau\left(a'\middle|r\right)$ and $t \in \mathbb{R}$, we have:*

$$b + tb' \in \mathsf{t}_\tau\left(a + ta'\middle|(1+|t|)r\right);$$

2. *consequently,*

$$\sup\left\{\left\|b - b'\right\|_{\mathfrak{B}} : b \in \mathsf{t}_\tau(a|r), b' \in \mathsf{t}_\tau\left(a'\middle|r\right)\right\} \leqslant \left\|a - a'\right\|_{\mathfrak{A}} + 2r\chi(\tau).$$

3. *In particular:*
$$\mathrm{diam}\left(\mathfrak{t}_{\tau}\left(a|r\right),\|\cdot\|_{\mathfrak{B}}\right)\leqslant 2r\chi\left(\tau\right).$$

Proof. Let $b\in\mathfrak{t}_{\tau}\left(a|r\right)$, $b'\in\mathfrak{t}_{\tau}\left(a'\middle|r\right)$ and $t\in\mathbb{R}$. By Definition (2.3.3), there exists $d\in\mathfrak{sa}\left(\mathfrak{D}\right)$ such that $\pi_{\mathfrak{B}}\left(d\right)=b$, $\pi_{\mathfrak{A}}\left(d\right)=a$, $\mathsf{L}_{\mathfrak{D}}\left(d\right)\leqslant r$. Similarly, there exists $d'\in\mathfrak{sa}\left(\mathfrak{D}\right)$ such that $\pi_{\mathfrak{A}}\left(d'\right)=a'$, $\pi_{\mathfrak{B}}\left(d'\right)=b'$ and $\mathsf{L}_{\mathfrak{D}}\left(d'\right)\leqslant r$. Then:

$$\mathsf{L}_{\mathfrak{D}}\left(d+td'\right)\leqslant\mathsf{L}_{\mathfrak{D}}\left(d\right)+|t|\mathsf{L}_{\mathfrak{D}}\left(d'\right)\leqslant r+|t|r,$$

and $\pi_{\mathfrak{A}}\left(d+td'\right)=a+ta'$, so $b+tb'=\pi_{\mathfrak{B}}\left(d+td'\right)\in\mathfrak{t}_{\tau}\left(a+ta'\middle|r(1+|t|)\right)$ by Definition (2.3.3). This completes the proof of (1).

Now, let $a,a'\in\mathfrak{sa}\left(\mathfrak{A}\right)$ and $r\geqslant\max\{\mathsf{L}_{\mathfrak{A}}\left(a\right),\mathsf{L}_{\mathfrak{A}}\left(a'\right)\}$. Then if $b\in\mathfrak{t}_{\tau}\left(a|r\right)$ and $b'\in\mathfrak{t}_{\tau}\left(a'\middle|r\right)$ then $b-b'\in\mathfrak{t}_{\tau}\left(a-a'\middle|2r\right)$ by the proof of (1). By Theorem (2.3.5), we have:

$$\left\|b-b'\right\|_{\mathfrak{B}}\leqslant\left\|a-a'\right\|_{\mathfrak{A}}+2r\chi\left(\tau\right).\tag{2.3.1}$$

This proves Assertion (2) of our proposition.

Assertion (3) is now obtained from Inequality (2.3.1) with $a=a'$. This completes our proof. □

Our next goal is to relate the multiplicative structure of Leibniz quantum compact metric spaces with target sets of tunnels. The following property makes explicit use of the Leibniz property of Definition (1.1.1).

Theorem 2.3.9. *Let $\left(\mathfrak{A},\mathsf{L}_{\mathfrak{A}}\right)$ and $\left(\mathfrak{B},\mathsf{L}_{\mathfrak{B}}\right)$ be two K-quantum compact metric spaces, for some $K\geqslant 1$, and let $\tau:\left(\mathfrak{A},\mathsf{L}_{\mathfrak{A}}\right)\xleftarrow{\pi_{\mathfrak{A}}}\left(\mathfrak{D},\mathsf{L}_{\mathfrak{D}}\right)\xrightarrow{\pi_{\mathfrak{B}}}\left(\mathfrak{B},\mathsf{L}_{\mathfrak{B}}\right)$ be a K-tunnel from $\left(\mathfrak{A},\mathsf{L}_{\mathfrak{A}}\right)$ to $\left(\mathfrak{B},\mathsf{L}_{\mathfrak{B}}\right)$.*

Let $a,a'\in\mathrm{dom}\left(\mathsf{L}_{\mathfrak{A}}\right)$, and let $l\geqslant\max\{\mathsf{L}_{\mathfrak{A}}\left(a\right),\mathsf{L}_{\mathfrak{A}}\left(a'\right)\}$. Let

$$C:=Kl\left(\|a\|_{\mathfrak{A}}+\|a\|_{\mathfrak{A}'}+l+2\chi\left(\tau\right)\right).$$

If $b\in\mathfrak{t}_{\tau}\left(a|l\right)$ and $b'\in\mathfrak{t}_{\gamma}\left(a'\middle|l\right)$, then:

$$\Re\left(bb'\right)\in\mathfrak{t}_{\tau}\left(\Re\left(aa'\right)\middle|C\right),$$

and:

$$\Im\left(bb'\right)\in\mathfrak{t}_{\tau}\left(\Im\left(aa'\right)\middle|C\right).$$

Proof. Let $a,a'\in\mathfrak{sa}\left(\mathfrak{A}\right)$, $r\geqslant\max\{\mathsf{L}_{\mathfrak{A}}\left(a\right),\mathsf{L}_{\mathfrak{A}}\left(a'\right)\}$. Let $b\in\mathfrak{t}_{\gamma}\left(a|l\right)$, $b'\in\mathfrak{t}_{\gamma}\left(a'\middle|l\right)$. Let $d,d'\in\mathfrak{sa}\left(\mathfrak{D}\right)$ such that:

$$\mathsf{L}_{\mathfrak{D}}\left(d\right)\leqslant l,\quad\pi_{\mathfrak{A}}\left(d\right)=a,\quad\pi_{\mathfrak{B}}\left(d\right)=b$$

and

$$\mathsf{L}_{\mathfrak{D}}\left(d'\right)\leqslant l,\quad\pi_{\mathfrak{A}}\left(d'\right)=a',\quad\pi_{\mathfrak{B}}\left(d'\right)=b'.$$

By Theorem (2.3.5), we have:

$$\|d\|_{\mathfrak{D}}\leqslant\|a\|_{\mathfrak{A}}+l\chi\left(\tau\right)\text{ and }\left\|d'\right\|_{\mathfrak{D}}\leqslant\left\|a'\right\|_{\mathfrak{A}}+l\chi\left(\tau\right).$$

Since $(\mathfrak{D}, \mathsf{L}_\mathfrak{D})$ is a K-quantum compact metric space, we get:

$$\begin{aligned}\mathsf{L}_\mathfrak{D}(\Re(dd')) &\leqslant K(\|d\|_\mathfrak{D}\mathsf{L}_\mathfrak{D}(d') + \mathsf{L}_\mathfrak{D}(d)\left\|d'\right\|_\mathfrak{D} + \mathsf{L}_\mathfrak{D}(d')\mathsf{L}_\mathfrak{D}(d)) \\ &\leqslant K\left((\|a\|_\mathfrak{A} + l\chi(\tau))l + l(\left\|a'\right\|_\mathfrak{A} + l\chi(\tau)) + l^2\right) = C.\end{aligned}$$

Since $\pi_\mathfrak{A}(\Re(dd')) = \Re(aa')$, we conclude our theorem holds true for the Jordan product.

The proof for the Lie product is similar. □

2.4 THE PROPINQUITY

Definition 2.4.1. The *K-propinquity* between two K-quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ and $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$, is the nonnegative number:

$$\Lambda^*_K((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{B}, \mathsf{L}_\mathfrak{B})) := \inf\left\{\chi(\tau) : \tau \ K\text{-tunnel from } (\mathfrak{A}, \mathsf{L}_\mathfrak{A}) \text{ to } (\mathfrak{B}, \mathsf{L}_\mathfrak{B})\right\}.$$

We will use the following definition in this paper. For a class C and an equivalence relation $\sim$ on C, a function d on $C \times C$ is called a *metric up to* $\sim$ (or a *pseudo-metric*, in short) if the following three properties hold:

1. $\forall x, y \in C, \quad d(x, y) = 0$ if and only if $x \sim y$,
2. $\forall x, y \in C, \quad d(x, y) = d(y, x)$,
3. $\forall x, y, z \in C, \quad d(x, z) \leqslant d(x, y) + d(y, z)$.

Proposition 2.4.2. *The propinquity is a pseudo-metric on the class of K-quantum compact metric spaces, such that:*

1. *if $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ and $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$ are fully quantum isometric, then $\Lambda^*_K((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{B}, \mathsf{L}_\mathfrak{B})) = 0$,*
2. *for any two K-quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ and $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$,*

$$\Lambda^*_K((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{B}, \mathsf{L}_\mathfrak{B})) \leqslant \max\left\{\mathrm{qdiam}(\mathfrak{A}, \mathsf{L}_\mathfrak{A}), \mathrm{qdiam}(\mathfrak{B}, \mathsf{L}_\mathfrak{B})\right\}.$$

Proof. Let $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ and $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$ be two K-quantum compact metric spaces. By Lemma (2.2.11), there exists a K-tunnel τ from $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ to $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$ whose extent $\chi(\tau)$ is at most $\max\{\mathrm{qdiam}(\mathfrak{A}, \mathsf{L}_\mathfrak{A}), \mathrm{qdiam}(\mathfrak{B}, \mathsf{L}_\mathfrak{B})\}$. By Definition (2.4.1), we thus conclude that $\Lambda^*((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{B}, \mathsf{L}_\mathfrak{B})) \leqslant \max\{\mathrm{qdiam}(\mathfrak{A}, \mathsf{L}_\mathfrak{A}), \mathrm{qdiam}(\mathfrak{B}, \mathsf{L}_\mathfrak{B})\}$. We also note trivially that $\Lambda^*_K((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{B}, \mathsf{L}_\mathfrak{B})) \geqslant 0$.

Let now $(\mathfrak{D}, \mathsf{L}_\mathfrak{D})$ be some other K-quantum compact metric space. Let $\varepsilon > 0$. By Definition (2.4.1), there exists a K-tunnel τ_1 from $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ to $(\mathfrak{D}, \mathsf{L}_\mathfrak{D})$ such that $\chi(\tau_1) < \Lambda^*_K((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{D}, \mathsf{L}_\mathfrak{D})) + \frac{\varepsilon}{3}$. Moreover, there exists a K-tunnel τ_2 from $(\mathfrak{D}, \mathsf{L}_\mathfrak{D})$

to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ such that $\chi(\tau_2) < \Lambda^*_K((\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) + \frac{\varepsilon}{3}$. By Theorem (2.3.1), there exists a K-tunnel τ from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ with $\chi(\tau) \leqslant \chi(\tau_1) + \chi(\tau_2) + \frac{\varepsilon}{3}$. Therefore:

$$\begin{aligned}
\Lambda^*_K((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) &\leqslant \chi(\tau) \\
&\leqslant \chi(\tau_1) + \chi(\tau_2) + \frac{\varepsilon}{3} \\
&\leqslant \Lambda^*_K((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})) + \frac{\varepsilon}{3} + \Lambda^*_K((\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \Lambda^*_K((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})) + \Lambda^*_K((\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) + \varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we conclude that

$$\Lambda^*_K((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \Lambda^*_K((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})) + \Lambda^*_K((\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})).$$

Since $\chi(\tau) = \chi\left(\tau^{-1}\right)$, it is immediate that

$$\Lambda^*_K((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = \Lambda^*_K((\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}), (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})).$$

Therefore, Λ^*_K is a pseudo-metric on the class of K-quantum compact metric spaces.

Now, assume that there exists a full quantum isometry from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. Let $\tau : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xleftarrow{\mathrm{id}_{\mathfrak{A}}} (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xrightarrow{\pi} (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. By Definition (2.2.1), τ is indeed a K-tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$, whose extent is trivially 0. So $0 \leqslant \Lambda^*((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (B, \mathsf{L}_{\mathfrak{B}})) \leqslant 0$, as needed. □

We now turn to actually fully characterizing what zero propinquity means.

Theorem 2.4.3. *Fix $K \geqslant 1$. For any two K-quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$,*

$$\Lambda^*_K((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$$

if, and only if, $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are fully quantum isometric.

Proof. By Proposition (2.4.2), we already know that our condition is sufficient. We henceforth assume that $\Lambda^*_K((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ for two fixed K-quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

For every $n \in \mathbb{N}$, there exists, by Definition (2.4.1), a K-tunnel τ_n from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$, with extent $\chi(\tau_n) \leqslant \frac{1}{n+1}$. We will henceforth simplify our notation somewhat, and denote $\mathfrak{t}_n(\cdot|\cdot)$ in place of $\mathfrak{t}_{\tau_n}(\cdot|\cdot)$ for all $n \in \mathbb{N}$.

We present our proof as a succession of claims, followed by their own proof, to expose the main structure of our argument.

As our first step, given a fixed $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$, we show how to extract a potential image for a in $\mathfrak{B}$ from our target sets $\mathfrak{T}_{\cdot}(a|\cdot)$, using a compactness argument.

Claim 2.4.4. *For any $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$, and for any strictly increasing function $f : \mathbb{N} \to \mathbb{N}$, there exists a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ and $\pi(a) \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ such that, for all $r \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, the sequence*

$$\left(\mathfrak{t}_{f\circ g(n)}(a|r)\right)_{n\in\mathbb{N}}$$

converges to the singleton $\{\pi(a)\}$ *in the Hausdorff distance* $\mathsf{Haus}[\mathfrak{B}]$ *induced by* $\|\cdot\|_{\mathfrak{B}}$ *on the compact subsets of* $\mathfrak{B}$*. Moreover, for any sequence* $(b_m)_{m\in\mathbb{N}}$ *in* $\mathfrak{B}$*, we have the following property:*

$$\left(\forall m \in \mathbb{N} \quad b_m \in \mathfrak{t}_{f\circ g(m)}(a|r)\right) \implies \lim_{m\to\infty} b_m = \pi(a).$$

Fix $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and let $r \geqslant \mathsf{L}_{\mathfrak{A}}(a)$. Let $f : \mathbb{N} \to \mathbb{N}$ be any strictly increasing function. Let $\varepsilon \in (0,1)$. By Proposition (2.3.5), for all $n \in \mathbb{N}$, the set $\mathfrak{t}_n(a|r)$ is a subset of the set:

$$\mathfrak{c}(a,r) := \{b \in \mathfrak{sa}(\mathfrak{B}) : \|b\|_{\mathfrak{B}} + \mathsf{L}_{\mathfrak{B}}(b) \leqslant (2r + \|a\|_{\mathfrak{A}}) + r\}.$$

By Theorem (1.1.22), the set $\mathfrak{c}(a,r)$ is compact in $(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$. In particular, the set $\mathfrak{c}(a,r)$ is complete for $\|\cdot\|_{\mathfrak{B}}$. Therefore, the Hausdorff distance $\mathsf{Haus}[\mathfrak{B}]$ also induces a compact topology on the set of all compact subsets of $\mathfrak{c}(a,r)$. By Proposition (2.3.6), the sets $\mathfrak{t}_n(a|r)$ are indeed compact subsets of $\mathfrak{c}(a,r)$ for all $n \in \mathbb{N}$. Therefore, the sequence $\left(\mathfrak{t}_{f(n)}(a|r)\right)_{n\in\mathbb{N}}$ has a convergent subsequence $\left(\mathfrak{t}_{f(g(n))}(a|r)\right)_{n\in\mathbb{N}}$ for the Hausdorff distance $\mathsf{Haus}[\mathfrak{B}]$, with limit a nonempty compact subset of $\mathfrak{c}(a,r)$; Let $S_{fcircg}(a,r)$ denote this limit.

We now prove that $S_{f\circ g}(a,r)$ is a singleton. Again, by Theorem (2.3.8), we conclude that, since the diameter is a continuous function with respect to the Hausdorff distance,

$$0 \leqslant \mathrm{diam}\left(S_{f\circ g}(a,r), \|\cdot\|_{\mathfrak{B}}\right) = \lim_{n\to\infty} \mathrm{diam}\left(\mathfrak{t}_{f(g(n))}(a|r), \|\cdot\|_{\mathfrak{B}}\right) \tag{2.4.1}$$

$$\leqslant \limsup_{n\to\infty} 2r\chi\left(\tau_{f(g(n))}\right) = \limsup_{n\to\infty} \frac{2r}{f(g(n))+1} = 0. \tag{2.4.2}$$

So $S_{f\circ g}(a,r)$ is non-empty compact set with diameter 0, so it is indeed a singleton.

Our reasoning so far applies to the special case when $r = \mathsf{L}_{\mathfrak{A}}(a)$. We denote the limit of $(\mathfrak{t}_{f(g(n))}(a|\mathsf{L}_{\mathfrak{A}}(a)))_{n\in\mathbb{N}}$ by $\{\pi(a)\}$. Now, if $r > \mathsf{L}_{\mathfrak{A}}(a)$, then for all $n \in \mathbb{N}$, we have $\mathfrak{t}_{f(g(n))}(a|\mathsf{L}_{\mathfrak{A}}(a)) \subseteq \mathfrak{t}_{f(g(n))}(a|r)$, and therefore,

$$\begin{aligned}
\mathsf{Haus}[\mathfrak{B}](\mathfrak{t}_{f(g(n))}(a|r), \{\pi(a)\}) &\leqslant \mathsf{Haus}[\mathfrak{B}](\mathfrak{t}_{f(g(n))}(a|r), \mathfrak{t}_{f(g(n))}(a|\mathsf{L}_{\mathfrak{A}})) \\
&\quad + \mathsf{Haus}[\mathfrak{B}](\mathfrak{t}_{f(g(n))}(a|\mathsf{L}_{\mathfrak{A}}(a)), \{\pi(a)\}) \\
&\leqslant \mathrm{diam}\left(\mathfrak{t}_{f(g(n))}(a|r), \|\cdot\|_{\mathfrak{B}}\right) \\
&\quad + \mathsf{Haus}[\mathfrak{B}](\mathfrak{t}_{f(g(n))}(a|\mathsf{L}_{\mathfrak{A}}(a)), \{\pi(a)\}) \\
&\leqslant \frac{2r}{f(g(n))+1} + \mathsf{Haus}[\mathfrak{B}](\mathfrak{t}_{f(g(n))}(a|\mathsf{L}_{\mathfrak{A}}(a)), \{\pi(a)\}) \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

So indeed, for all $r \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, the sequence $(\mathfrak{t}_{f(g(n))}(a|r))_{n\in\mathbb{N}}$ converges to $\{\pi(a)\}$ for $\mathsf{Haus}[\mathfrak{B}]$.

We return to the general case with $r \geqslant \mathsf{L}_{\mathfrak{A}}(a)$. For each $n \in \mathbb{N}$, let $b_n \in \mathfrak{t}_{f\circ g(n)}(a|r)$ be chosen arbitrarily (which is possible since $\mathfrak{t}_{f\circ g(n)}(a|r)$ is not empty by Lemma (2.3.4)). Then Proposition (2.3.8) proves that $(b_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{sa}(\mathfrak{B})$

for the norm of $\mathfrak{B}$. Indeed, let $\varepsilon > 0$. Then there exists $M \in \mathbb{N}$ such that for all $n \geqslant M$, we have $\frac{2r}{n+1} < \frac{\varepsilon}{2}$, and therefore, by Theorem (2.3.8),

$$\operatorname{diam}\left(\mathfrak{t}_{f\circ g(m)}(a|r), \|\cdot\|_{\mathfrak{B}}\right) < \frac{1}{2}\varepsilon. \tag{2.4.3}$$

By definition of the function g, there also exists $M' \in \mathbb{N}$ such that, for all $p, q \geqslant M'$, we have:

$$\mathsf{Haus}[\mathfrak{B}]\left(\mathfrak{t}_{f\circ g(p)}(a|r), \mathfrak{t}_{f\circ g(q)}(a|r)\right) < \frac{1}{2}\varepsilon, \tag{2.4.4}$$

since $\left(\mathfrak{t}_{f\circ g(n)}(a|r)\right)_{n\in\mathbb{N}}$ converges, and hence is Cauchy, for $\mathsf{Haus}[\mathfrak{B}]$.

Let $p, q \geqslant \max\{M, M'\}$. By Inequality (2.4.4), there exists:

$$c_q \in \mathfrak{t}_{f\circ g(q)}(a|r)$$

such that $\left\|c_q - b_p\right\|_{\mathfrak{B}} \leqslant \frac{1}{2}\varepsilon$. By Inequality (2.4.3), we also have $\left\|b_q - c_q\right\|_{\mathfrak{B}} \leqslant \frac{1}{2}\varepsilon$. Hence $\left\|b_p - b_q\right\|_{\mathfrak{B}} \leqslant \varepsilon$. Thus $(b_n)_{n\in\mathbb{N}}$ is indeed a Cauchy sequence in $\mathfrak{sa}(\mathfrak{B})$. Since $\mathfrak{sa}(\mathfrak{B})$ is complete, the sequence $(b_n)_{n\in\mathbb{N}}$ converges. Let us temporarily denote its limit by b.

It is easy to check that $b \in S_{f\circ g}(a, r)$: for any $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that for all $m \geqslant M$, the diameter of $\mathfrak{t}_{f\circ g(m)}(a|r)$ is less than $\frac{1}{2}\varepsilon$, and there exists $M' \in \mathbb{N}$ such that for all $m \geqslant M'$, we have
$normb_m - b\mathfrak{B} < \frac{1}{2}\varepsilon$, so for all $m \geqslant \max\{M, M'\}$, the set $\mathfrak{t}_{f\circ g(m)}(a|r)$ lies within the open ε-neighborhood of b for $\|\cdot\|_{\mathfrak{B}}$. So $b \in \{\pi(a)\}$ by definition of the Hausdorff distance $\mathsf{Haus}[\mathfrak{B}]$. Hence $b = \pi(a)$, as desired.

Moreover, if $b_n \in \mathfrak{t}_{f(g(n))}(a|\mathsf{L}_{\mathfrak{A}}(a))$, then $\mathsf{L}(b_n) \leqslant \mathsf{L}_{\mathfrak{A}}(a)$ for all $n \in \mathbb{N}$, so by lower semicontinuity of $\mathsf{L}_{\mathfrak{B}}$, we also conclude that $\mathsf{L}(b) \leqslant \mathsf{L}_{\mathfrak{A}}(a)$.

This completes the proof of our claim.

Our next step is to choose images for all elements in $\operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$ *in a coherent fashion, based upon Claim (2.4.4), and a diagonal argument:*

Claim 2.4.5. *There exists an increasing function $f : \mathbb{N} \to \mathbb{N}_+$ and a function $\pi : \operatorname{dom}(\mathsf{L}_{\mathfrak{A}}) \to \operatorname{dom}(\mathsf{L}_{\mathfrak{B}})$ such that, for any $a \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$ and for any $r \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, we have:*

$$\lim_{n\to\infty} \mathsf{Haus}[\mathfrak{B}]\left(\mathfrak{t}_{f(n)}(a|r), \{\pi(a)\}\right) = 0.$$

Moreover, for any $a \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$, any sequence $(b_m)_{m\in\mathbb{N}}$ in $\mathfrak{B}$ and any $r \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, we have:

$$\left(\forall m \in \mathbb{N} \quad b_m \in \mathfrak{T}_{\Gamma_{f(m)^{-1}}}(a|r)\right) \implies \lim_{m\to\infty} b_m = \pi(a).$$

Let $\mathfrak{a} = \{a_k : k \in \mathbb{N}\}$ be a countable, dense subset of $\operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$. To ease notations, let $l_k = \mathsf{L}_{\mathfrak{A}}(a_k)$ for all $k \in \mathbb{N}$. By Claim (2.4.4), for each $k \in \mathbb{N}$, there exists a strictly increasing function $g_k : \mathbb{N} \to \mathbb{N}$, such that, for all $r \geqslant l_k$, the sequence $\left(\mathfrak{t}_{g_k(m)}(a_k|r)\right)_{n\in\mathbb{N}}$ converges to some singleton $\{\pi(a_k)\}$ in $\mathfrak{B}$ for $\mathsf{Haus}[\mathfrak{B}]$. One then easily checks that, setting:

$$f : n \in \mathbb{N} \longmapsto g_0 \circ g_1 \circ \ldots g_n(n),$$

for all $k \in \mathbb{N}$ and for all $r \geqslant l_k$, the sequence $\left(\mathfrak{t}_{f(m)}(a_k|r)\right)_{n\in\mathbb{N}}$ converges in $\mathsf{Haus}[\mathfrak{B}]$ to $\{\pi(a_k)\}$ for all $k \in \mathbb{N}$. Moreover, by Claim (2.4.4), for all $k \in \mathbb{N}$, for all $r \geqslant l_k$, and for sequences $(b_m)_{m\in\mathbb{N}} \in \mathfrak{sa}(\mathfrak{B})$ with $b_n \in \mathfrak{t}_{f(n)}(a|r)$ for all $n \in \mathbb{N}$,

$$\lim_{m\to\infty} b_m = \pi(a). \tag{2.4.5}$$

We now extend π *thus defined to all of* $\operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$.

Let $a \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$ be chosen. Let $\varepsilon > 0$. There exists $a' \in \mathfrak{a}$ with $\|a - a'\|_{\mathfrak{A}} < \frac{\varepsilon}{6}$. Let $r \geqslant \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{A}}(a')\}$. Since $\left(\mathfrak{t}_{f(n)}\left(a'\middle|r\right)\right)_{n\in\mathbb{N}}$ converges (to $\{\pi'(a)\}$), it is Cauchy. Let $M \in \mathbb{N}$ such that, for all $p, q \geqslant M$, we have:

$$\mathsf{Haus}[\mathfrak{B}]\left(\mathfrak{t}_{f(p)}\left(a'\middle|r\right), \mathfrak{t}_{f(q)}\left(a'\middle|r\right)\right) < \frac{\varepsilon}{3}.$$

Let $M' \in \mathbb{N}$ such that for all $m \geqslant M'$, we have $\frac{1}{f(m)+1} < \frac{\varepsilon}{12r}$. Let $m \geqslant \max\{M, M'\}$. For all $b \in \mathfrak{t}_{f(m)}(a|r)$ and $b' \in \mathfrak{t}_{f(m)}\left(a'\middle|r\right)$, by Theorem (2.3.8), we have

$$\left\|b - b'\right\|_{\mathfrak{B}} \leqslant \left\|a - a'\right\|_{\mathfrak{A}} + \frac{2r}{f(m)+1} \leqslant \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.$$

Therefore, for all $m \geqslant M'$, we conclude that $\mathsf{Haus}[\mathfrak{B}](\mathfrak{t}_{f(n)}(a|r), \mathfrak{t}_{f(n)}\left(a'\middle|r\right)) < \frac{\varepsilon}{3}$. Hence, for all $p, q \geqslant \max\{M, M'\}$, we obtain:

$$\begin{aligned}\mathsf{Haus}[\mathfrak{B}](\mathfrak{t}_{f(p)}(a|r), \mathfrak{t}_{f(q)}(a|r)) &\leqslant \mathsf{Haus}[\mathfrak{B}](\mathfrak{t}_{f(p)}(a|r), \mathfrak{t}_{f(p)}\left(a'\middle|r\right)) \\ &\quad + \mathsf{Haus}[\mathfrak{B}](\mathfrak{t}_{f(p)}\left(a'\middle|r\right), \mathfrak{t}_{f(q)}\left(a'\middle|r\right)) \\ &\quad + \mathsf{Haus}[\mathfrak{B}](\mathfrak{t}_{f(q)}\left(a'\middle|r\right), \mathfrak{t}_{f(q)}(a|r)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.\end{aligned}$$

So the sequence $(\mathfrak{t}_{f(n)}(a|r))_{n\in\mathbb{N}}$ is a Cauchy sequence for $\mathsf{Haus}[\mathfrak{B}]$. Since $\mathfrak{B}$ is complete, so is $\mathsf{Haus}[\mathfrak{B}]$, and thus the sequence $(\mathfrak{t}_{f(n)}(a|r))_{n\in\mathbb{N}}$ converges for $\mathsf{Haus}[\mathfrak{B}]$. Now, following Claim (2.4.4), the limit of $(\mathfrak{t}_{f(n)}(a|r))_{n\in\mathbb{N}}$ is a singleton consisting of the limit of any sequence $(b_m)_{m\in\mathbb{N}}$ chosen so that $b_m \in \mathfrak{t}_{f(n)}(a|r)$ for all $m \in \mathbb{N}$ and for all $r \geqslant \mathsf{L}_{\mathfrak{A}}(a)$. We denote this singleton by $\{\pi(a)\}$. Note, in particular, that by lower semicontinuity, $\mathsf{L}_{\mathfrak{B}}(\pi(a)) \leqslant \mathsf{L}_{\mathfrak{A}}(a)$ again.

We thus have defined a map $\pi : \operatorname{dom}(\mathsf{L}_{\mathfrak{A}}) \to \operatorname{dom}(\mathsf{L}_{\mathfrak{B}})$. *We now enter the third phase of our construction of* π*, where we establish algebraic properties from the properties of target sets.*

Claim 2.4.6. *The map* $\pi : \operatorname{dom}(\mathsf{L}_{\mathfrak{A}}) \to \operatorname{dom}(\mathsf{L}_{\mathfrak{B}})$ *is* $\mathbb{R}$*-linear and such that* $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant \mathsf{L}_{\mathfrak{A}}$ *on* $\operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$*. Moreover, h has norm at most one and thus can be extended to an* $\mathbb{R}$*-linear map, still denoted by* π*, from* $\mathfrak{sa}(\mathfrak{A})$ *to* $\mathfrak{sa}(\mathfrak{B})$*, of norm at most one.*

Let $a, a' \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$ and $t \in \mathbb{R}$. Let $r \geqslant \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{A}}(a')\}$. For all $m \in \mathbb{N}$, we let $b_m \in \mathfrak{t}_{f(m)}(a|r)$ and $b'_m \in \mathfrak{t}_{f(m)}\left(a'\middle|r\right)$. By Claim (2.4.5), we have $\lim_{m\to\infty} b_m = \pi(a)$ and $\lim_{m\to\infty} b'_m = \pi(a')$.

Now, by Theorem (2.3.8), $b_m + tb'_m \in \mathfrak{t}_{f(m)}\left(a + ta'\middle|(1+|t|)r\right)$ for all $m \in \mathbb{N}$. Since:

$$\mathsf{L}_{\mathfrak{A}}(a + ta') \leqslant \mathsf{L}_{\mathfrak{A}}(a) + |t|\mathsf{L}_{\mathfrak{A}}(a') \leqslant (1+|t|)r$$

by construction, we conclude from Claim (2.4.5) that:

$$\pi(a + ta') = \lim_{m\to\infty}(b_m + tb'_m) = \lim_{m\to\infty} b_m + t\lim_{m\to\infty} b'_m = \pi(a) + t\pi(a'),$$

as desired. *Hence, π is linear on* $\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$.

Now, let $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and let $(b_m)_{m\in\mathbb{N}}$ be a sequence in $\mathfrak{sa}(\mathfrak{B})$ such that for all $m \in \mathbb{N}$, we have $b_m \in \mathfrak{t}_{f(m)}(a|\mathsf{L}_{\mathfrak{A}}(a))$. Since $\mathsf{L}_{\mathfrak{B}}$ is lower-semi-continuous by assumption, we have:

$$\mathsf{L}_{\mathfrak{B}}(\pi(a)) \leqslant \liminf_{m\to\infty} \mathsf{L}_{\mathfrak{B}}(b_m) \leqslant \mathsf{L}_{\mathfrak{A}}(a).$$

Moreover, Theorem (2.3.5), we can prove:

$$\|\pi(a)\|_{\mathfrak{B}} = \lim_{m\to\infty}\|b_m\|_{\mathfrak{B}} \leqslant \lim_{m\to\infty}\left(\|a\|_{\mathfrak{A}} + 2\frac{\mathsf{L}_{\mathfrak{A}}(a)}{f(m)+1}\right) = \|a\|_{\mathfrak{A}}. \tag{2.4.6}$$

Hence, π is a uniformly continuous linear map from $\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$, and thus it extends uniquely to a continuous linear map from $\mathfrak{sa}(\mathfrak{A})$ to $\mathfrak{sa}(\mathfrak{B})$, which we still denote by π. We note that the norm of π is, at most, one.

We now turn to the multiplicative properties of π.

Claim 2.4.7. *The map $\pi : \mathfrak{sa}(\mathfrak{A}) \to \mathfrak{sa}(\mathfrak{B})$ is a unital Jordan-Lie algebra homomorphism of norm* 1.

Let $a, a' \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and $r \geqslant \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{A}}(a')\}$. Let

$$C := Kr(\|a\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}'} + r + 2\chi(\tau)).$$

Let $m \in \mathbb{N}$ and choose $b_m \in \mathfrak{t}_{f(m)}(a|r)$ and $b'_m \in \mathfrak{t}_{f(m)}\left(a'\middle|r\right)$. By Theorem (2.3.9), we have

$$\Re(b_m b'_m) \in \mathfrak{t}_{f(m)}\left(\Re(aa')\middle|C\right).$$

Thus, we conclude by Claim (2.4.5) that:

$$\pi(\Re(aa')) = \lim_{m\to\infty}\Re(b_m b'_m) = \Re(\lim_{m\to\infty} b_m \lim_{m\to\infty} b'_m) = \Re(\pi(a)\pi(a')).$$

Similarly, we would prove that $\pi(\Im(aa')) = \Im(\pi(a)\pi(a'))$.

From the construction of π, since $1_{\mathfrak{B}} \in \mathfrak{t}_{f(n)}(1_{\mathfrak{A}}|r)$ for any $n \in \mathbb{N}$ and $r \geqslant 0$, we conclude easily that $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$. Since π is a linear map of norm at most 1 by Claim (2.4.6), we conclude that π has norm one.

Thus $\pi : \mathfrak{sa}(\mathfrak{A}) \to \mathfrak{sa}(\mathfrak{B})$ is a Jordan-Lie homomorphism of norm 1*, such that* $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant \mathsf{L}_{\mathfrak{A}}$ *on* $\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$.

Claim 2.4.8. *The continuous unital Jordan-Lie homomorphism $\pi : \mathfrak{sa}(\mathfrak{A}) \to \mathfrak{sa}(\mathfrak{B})$ extends uniquely to a unital *-homomorphism $\mathfrak{A} \to \mathfrak{B}$, still denoted by π.*

We set, for all $a \in \mathfrak{A}$:

$$\pi(a) = \pi(\Re(a)) + i\pi(\Im(a)),$$

and we trivially check that this definition of π is consistent on $\mathfrak{sa}(\mathfrak{A})$. Moreover, it is straightforward that π thus extended is a continuous linear map on $\mathfrak{A}$, with values in $\mathfrak{B}$. Moreover, by construction, $\pi(a^*) = \pi(a)^*$ for all $a \in \mathfrak{A}$. So π is a $*$-preserving linear map on $\mathfrak{A}$.

We will write, for any $a, b \in \mathfrak{A}$, the Jordan product of a with b as $a \circ b := \frac{ab+ba}{2}$, and the Lie product as $\{a, b\} := \frac{ab-ba}{2i}$; these expressions are linear in each factor, but they are not usually self-adjoint unless $a, b \in \mathfrak{sa}(\mathfrak{A})$.

Let $a, b \in \mathfrak{A}$ be given. Using the fact that π, restricted to $\mathfrak{sa}(\mathfrak{A})$, is a homomorphism for the Jordan product by Claim (2.4.7), and π is linear on $\mathfrak{A}$, we have:

$$\begin{aligned}\pi(a \circ b) &= \pi(\Re a \circ \Re b) - \pi(\Im a \circ \Im b) + i(\pi(\Im a \circ \Re b) + \pi(\Re a \circ \Im b)) \\ &= \pi(\Re a) \circ \pi(\Re b) - \pi(\Im a) \circ \pi(\Im b) + i\pi(\Im a) \circ \pi(\Re b) + i\pi(\Re a) \circ \pi(\Im b) \quad (2.4.7) \\ &= \pi(a) \circ \pi(b).\end{aligned}$$

Again, the computation carries similarly to the Lie product.

To conclude, for all $a, b \in \mathfrak{A}$, by Equation (2.4.7) and its equivalent for the Lie product, as well as by linearity of π:

$$\pi(ab) = \pi(\Re(ab)) + i\pi(\Im ab) = \Re(\pi(a)\pi(b)) + i\Im(\pi(a)\pi(b)) = \pi(a)\pi(b).$$

*We have thus proven that π is a unital *-homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ with $\mathsf{L}_{\mathfrak{B}} \circ \pi \leqslant \mathsf{L}_{\mathfrak{A}}$ on* $\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$. This completes our construction of π. Now, we conclude our proof with the following last claim.

Claim 2.4.9. *The *-homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ is a *-isomorphism onto $\mathfrak{B}$, such that for all $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ we have $\mathsf{L}_{\mathfrak{B}}(\pi(a)) = \mathsf{L}_{\mathfrak{A}}(a)$.*

This last step of our proof consists in constructing the inverse of π using the same technique as used for the construction of π itself.

First, we apply Claims (2.4.4)–(2.4.8) to the sequence $(\tau^{-1}_{f(n)})_{n\in\mathbb{N}}$ of tunnels. We thus obtain that there exists a unital *-homomorphism $\psi : \mathfrak{B} \to \mathfrak{A}$ and a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that, for all $b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$, and for all $r \geqslant \mathsf{L}_{\mathfrak{B}}(b)$, we have

$$\lim_{n\to\infty} \mathsf{Haus}[\mathfrak{A}](\mathsf{t}_{\tau^{-1}_{f(g(n))}}(b|r), \{\psi(b)\}) = 0.$$

We claim that ψ is the inverse of π. To do so, let $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and $r \geqslant \mathsf{L}_{\mathfrak{A}}(a)$. Let $(b_m)_{m\in\mathbb{N}} \in \mathfrak{sa}(\mathfrak{B})^{\mathbb{N}}$ with $b_m \in \mathsf{t}_{f\circ g(m)}(a|r)$ for all $m \in \mathbb{N}$ and note that $\lim_{m\to\infty} b_m = \pi(a)$ by Claim (2.4.5). Similarly, let $(a_m)_{m\in\mathbb{N}} \in \mathfrak{sa}(\mathfrak{A})^{\mathbb{N}}$ such that for all $m \in \mathbb{N}$, we have $a_m \in \mathsf{t}_{\tau^{-1}_{f\circ g(m)}}(\pi(a)|r)$ (note that $r \geqslant \mathsf{L}_{\mathfrak{A}}(a) \geqslant \mathsf{L}_{\mathfrak{B}}(\pi(a))$). Again, we have $\lim_{m\to\infty} a_m = \psi(\pi(a))$. The key observation here is that by Definition (2.3.3), since $b_m \in \mathsf{t}_{f\circ g(m)}(a|r)$, we observe that $a \in \mathsf{t}_{\tau^{-1}_{f\circ g(m)}}(b_m|r)$.

Let $\varepsilon > 0$. Let $M \in \mathbb{N}$ such that for all $m \geqslant M$, we have $\|b_m - \pi(a)\|_{\mathfrak{B}} \leqslant \frac{1}{3}\varepsilon$. Let $M' \in \mathbb{N}$ be chosen so that for all $m \geqslant M'$ we have $\left\|a_m - \psi(\pi(a))\right\|_{\mathfrak{A}} < \frac{1}{3}\varepsilon$. Let $M'' \in \mathbb{N}$ such that $\frac{1}{1+f\circ g(m)} < \frac{\varepsilon}{12r}$ for all $m \geqslant M''$. Let $m \geqslant \max\{M, M', M''\}$. By Theorem (2.3.8), since $a \in \mathsf{t}_{\tau^{-1}_{(f\circ g(m))}}(b_m|r)$ and $a_m \in \mathsf{t}_{\tau^{-1}_{f\circ g(m)}}(\pi(a)|r)$, we have:

$$\|a - a_m\|_{\mathfrak{A}} \leqslant 2r\varepsilon + \|b_m - \pi(a)\|_{\mathfrak{B}} \leqslant \frac{2}{3}\varepsilon.$$

Hence:

$$\left\|a - \psi(\pi(a))\right\|_{\mathfrak{A}} \leqslant \|a - a_m\|_{\mathfrak{A}} + \left\|a_m - \psi(\pi(a))\right\|_{\mathfrak{A}} \leqslant \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we conclude that $a = \psi(\pi(a))$ for all $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$. Now, since $\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ is total in $\mathfrak{A}$ and π, ψ are *-homomorphisms, we conclude that:

$$\forall a \in \mathfrak{A} \quad a = \psi(\pi(a)).$$

Similarly, we would prove that for all $b \in \mathfrak{B}$, we have $b = \pi(\psi(b))$. Thus π *is a *-isomorphism from A onto* $\mathfrak{B}$. In particular, we conclude:

1. For all $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$, we have $\mathsf{L}_{\mathfrak{A}}(a) \geqslant \mathsf{L}_{\mathfrak{B}}(\pi(a)) \geqslant \mathsf{L}_{\mathfrak{A}}(\psi(\pi(a))) = \mathsf{L}_{\mathfrak{A}}(a)$, so $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$ on $\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$.
2. Similarly, $\mathsf{L}_{\mathfrak{A}} \circ \psi = \mathsf{L}_{\mathfrak{B}}$ and $\psi : \mathfrak{B} \to \mathfrak{A}$ is a *-isomorphism.

This concludes the proof of our main theorem. □

We have established that the K-propinquity is indeed a metric on the class of K-quantum compact metric spaces, up to full quantum isometry. We now explore its basic properties. We begin with the following, first result about continuity of the diameter.

Theorem 2.4.10. *If* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ *and* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ *are two* K*-quantum compact metric spaces, then*

$$\left|\mathrm{qdiam}(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) - \mathrm{qdiam}(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right| \leqslant 2\Lambda^*_K((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})).$$

Proof. Let $\tau : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xleftarrow{\pi_{\mathfrak{A}}} (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \xrightarrow{\pi_{\mathfrak{B}}} (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be a tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

Let $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ such that $\mathrm{qdiam}(\mathfrak{A}, \mathsf{L}) = \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}(\varphi, \psi)$. There exists $\varphi', \psi' \in \mathscr{S}(\mathfrak{B})$ such that $\mathsf{mk}_{\mathsf{T}}(\varphi \circ \pi_{\mathfrak{A}}, \varphi' \circ \pi_{\mathfrak{B}}) \leqslant \chi(\tau)$ and $\mathsf{mk}_{\mathsf{T}}(\psi \circ \pi_{\mathfrak{A}}, \psi' \circ \pi_{\mathfrak{B}}) < \chi(\tau)$.

We thus compute:

$$\begin{aligned}
&\mathrm{qdiam}(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) - \mathrm{qdiam}(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \\
&\quad = \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}(\varphi, \psi) - \mathrm{qdiam}(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \\
&\quad \leqslant \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}(\varphi, \psi) - \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}(\varphi', \psi') \\
&\quad = \mathsf{mk}_{\mathsf{T}}(\varphi \circ \pi_{\mathfrak{A}}, \psi \circ \pi_{\mathfrak{A}}) - \mathsf{mk}_{\mathsf{T}}(\varphi' \circ \pi_{\mathfrak{B}}, \psi' \circ \pi_{\mathfrak{B}}) \\
&\quad \leqslant \mathsf{mk}_{\mathsf{T}}(\varphi \circ \pi_{\mathfrak{A}}, \psi \circ \pi_{\mathfrak{B}}) - \mathsf{mk}_{\mathsf{T}}(\varphi' \circ \pi_{\mathfrak{B}}, \psi \circ \pi_{\mathfrak{A}}) + \mathsf{mk}_{\mathsf{T}}(\varphi' \circ \pi_{\mathfrak{B}}, \psi \circ \pi_{\mathfrak{A}}) \\
&\qquad - \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}}(\varphi' \circ \pi_{\mathfrak{B}}, \psi' \circ \pi_{\mathfrak{A}}) \\
&\quad \leqslant \mathsf{mk}_{\mathsf{T}}(\varphi \circ \pi_{\mathfrak{A}}, \varphi' \circ \pi_{\mathfrak{B}}) + \mathsf{mk}_{\mathsf{T}}(\psi \circ \pi_{\mathfrak{A}}, \psi' \circ \pi_{\mathfrak{B}}) \\
&\quad \leqslant 2\chi(\tau).
\end{aligned}$$

The result is symmetric in $\mathfrak{A}$ and $\mathfrak{B}$, so we have

$$\left|\mathrm{qdiam}\left(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}\right) - \mathrm{qdiam}\left(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}\right)\right| \leqslant 2\chi\left(\tau\right)$$

and thus, as τ was arbitrary, we conclude, as needed:

$$\left|\mathrm{qdiam}\left(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}\right) - \mathrm{qdiam}\left(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}\right)\right| \leqslant 2\Lambda_K^*\left(\left(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}\right), \left(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}\right)\right).$$

This proves our claim. □

The propinquity is also a complete metric. We refer to [36] for the proof of this fact. Completeness, in fact, was part of the motivation in the construction of the propinquity, which was introduced as the dual propinquity in [36], as a variation on the earlier construction of the so-called quantum propinquity [38], which we do not expect to be complete; the quantum propinquity can be obtained from our construction in these notes by restricting ourselves to tunnels constructed from bridges.

Theorem 2.4.11 ([36, Theorem 6.27])**.** *The K-propinquity Λ_K^* is complete, for any $K \geqslant 1$.*

We conclude with a brief observation. As proven in [4], if we return to Rieffel's notion of a quantum compact metric space as a pair $(\mathfrak{A}, \mathsf{L})$ where $\mathfrak{A}$ is now only an order unit space, thus also removing any Leibniz inequality from our definition, and if we define tunnels between such by asking for positive unital linear surjections in place of *-morphisms, then the construction above of the propinquity actually leads to a metric equivalent to Rieffel's original quantum Gromov-Hausdorff distance.

2.5 COMPACTNESS

Gromov proved a very useful theorem characterizing totally bounded subclasses of compact metric spaces for the Gromov-Hausdorff distance. We prove its analogue in noncommutative geometry. As our first step, we explicit a useful necessary condition, where a form of noncommutative covering number.

Definition 2.5.1. Let $K \geqslant 1$, and let A be a nonempty class of K-quantum compact metric spaces. Let $(\mathfrak{A}, \mathsf{L})$ be a K-quantum compact metric space and let $\varepsilon > 0$. The *covering number* $\mathrm{cov}_A\left(\mathfrak{A}, \mathsf{L}|\varepsilon\right)$ is:

$$\mathrm{cov}_{\mathfrak{A}}\left(\mathfrak{A}, \mathsf{L}|\varepsilon\right) = \inf\left\{\dim_{\mathbb{C}} \mathfrak{B} : \begin{array}{l} \exists(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in A \text{ such that} \\ \Lambda_K\left(\left(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}\right), \left(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}\right)\right) \leqslant \varepsilon \end{array}\right\}.$$

In particular, $\mathrm{cov}_K\left(\cdot|\cdot\right)$ is the covering number $\mathrm{cov}_{\mathrm{QCMS}_F}\left(\cdot|\cdot\right)$ where QCMS_K is the class of all K-quantum compact metric spaces.

Proposition 2.5.2. *Let A be a nonempty class of K-quantum compact metric spaces for some $K \geqslant 1$. If B is a subclass of the closure of $\{(\mathfrak{A}, \mathsf{L}) \in A : \dim \mathfrak{A} < \infty\}$ for the propinquity, and if B is totally bounded for Λ^*_K, then there exists $C > 0$ and $f : (0, \infty) \to \mathbb{N}$ such that, for all $(\mathfrak{A}, \mathsf{L}) \in B$:*

1. $\operatorname{diam}(\mathfrak{A}, \mathsf{L}) \leqslant C$,
2. $\forall \varepsilon > 0 \quad \operatorname{cov}_A(\mathfrak{A}, \mathsf{L}|\varepsilon) \leqslant f(n)$.

Proof. Since the function:

$$(\mathfrak{A}, \mathsf{L}) \in \mathscr{A} \mapsto \operatorname{qdiam}(\mathfrak{A}, \mathsf{L})$$

is 2-Lipschitz for the quantum Gromov-Hausdorff distance by Theorem (2.4.10), it is continuous, and thus it is bounded above since B is totally bounded.

Let $F := \{(\mathfrak{A}, \mathsf{L}) \in A : \dim \mathfrak{A} < \infty\}$. Now, let $\varepsilon > 0$. Since B is totally bounded and $B \subseteq \operatorname{cl}(F)$, there exists a finite subset B_ε of F which is $\frac{\varepsilon}{2}$-dense in B for Λ_K. Therefore, for each $(\mathfrak{A}, \mathsf{L}) \in B_\varepsilon$, there exists a finite dimensional K-quantum compact metric spaces $\mathfrak{f}(\mathfrak{A}, \mathsf{L}) \in F$ such that:

$$\Lambda_K((\mathfrak{A}, \mathsf{L}), \mathfrak{f}(\mathfrak{A}, \mathsf{L})) \leqslant \frac{\varepsilon}{2}$$

by assumption on A.

Let:

$$f(\varepsilon) := \max\left\{\dim_{\mathbb{C}} \mathfrak{f}(\mathfrak{A}, \mathsf{L}) : (\mathfrak{A}, \mathsf{L}) \in B\right\}.$$

If $(\mathfrak{A}, \mathsf{L}) \in B$, then there exists $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in B_\varepsilon$ such that $\Lambda_K((\mathfrak{A}, \mathsf{L}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) \leqslant \frac{\varepsilon}{2}$. By construction, $\operatorname{cov}_F(\mathfrak{A}, \mathsf{L}|\varepsilon) \leqslant f(\varepsilon)$. □

Our next step is to prove that certain classes of finite dimensional quantum compact metric spaces are indeed compact for the propinquity.

Theorem 2.5.3. *Let $\mathscr{QQCMS}_F$ be the class of all F-quantum compact metric spaces for $F \geqslant 1$. Let $d \in \mathbb{N} \setminus \{0\}$ and $K > 0$. The class:*

$$\mathscr{C}_{F,K,d} = \left\{(\mathfrak{A}, \mathsf{L}) \in \mathscr{QQCMS}_F : \dim_{\mathbb{C}} \mathfrak{A} \leqslant d \text{ and } \operatorname{qdiam}(\mathfrak{A}, \mathsf{L}) \leqslant K\right\}$$

is compact for the dual propinquity Λ_F.

Proof. Let $(\mathfrak{A}_n, \mathsf{L}_n)_{n \in \mathbb{N}}$ be a sequence in $\mathscr{C}_{F,K,d}$. Let $\mathfrak{M}_d$ be the C*-algebra of $d \times d$ matrices.

First, fix $n \in \mathbb{N}$. We identify the C*-algebra $\mathfrak{A}_n$ with a C*-subalgebra of $\mathfrak{M}_d$ as follows. Up to $*$-isomorphism, we can write $\mathfrak{A}_n = \oplus_{j \in J} \mathfrak{M}_{t(j)}$ where $J = \{1, \ldots, d\}$ and $t(1) \geqslant t(2) \geqslant \cdots \geqslant t(d)$. We note that t may be zero for some $j \in \{1, \ldots, d\}$, and the zero set of t is a tail of J.

Let $j \in \{1, \ldots, d\}$. Let $s(j) = \sum_{k=1}^{j} t(j)$ and set $s(0) = 0$. We now let Q_j be the projection given as the diagonal matrix whose only nonzero entries are 1 on the diagonal, from row $s(j-1)+1$ to $s(j)$, i.e. in block form:

$$Q_j = \begin{pmatrix} 0_{s(j-1)} & & \\ & 1_{s(j)-s(j-1)} & \\ & & 0_{d-s(j)} \end{pmatrix}.$$

Of course the projections Q_j are orthogonal and sum to the identity of $\mathfrak{M}_d$.

It then follows trivially that $\mathfrak{A}_n$ is isomorphic to $\mathfrak{A}'_n = \sum_{j\in J} Q_j \mathfrak{M}_d Q_j$. Let $\pi_n : \mathfrak{A}_n \to \mathfrak{A}'_n$ be the *-isomorpism thus constructed. Note that π_n is *not* a unital map from $\mathfrak{A}_n$ into $\mathfrak{M}_d$.

If 1_n is the unit of $\mathfrak{A}_n$ for all $n \in \mathbb{N}$, then $(\pi_n(1_n))_{n\in\mathbb{N}}$ is a sequence of diagonal projections, i.e. diagonal $d \times d$-matrices with entries in $\{0,1\}$. Thus, there exists a constant subsequence $(\pi_{g(n)}(1_{g(n)}))_{n\in\mathbb{N}}$, with value denoted by p, of $(\pi_n(1_n))_{n\in\mathbb{N}}$.

Let $\mathfrak{B} = p\mathfrak{M}_d p$. Note that for all $n \in \mathbb{N}$, the map $p\pi_{g(n)}p : \mathfrak{A}_n \to \mathfrak{B}$ is now a unital *-monomorphism. We shall henceforth omit the notation $\mathrm{ad}_p\pi_{g(n)}$ and simply identify $\mathfrak{A}_{g(n)}$ with $p\pi_{g(n)}(\mathfrak{A}_{g(n)})p$. We emphasize that with this identification, $1_{g(n)} = 1_{\mathfrak{B}}$.

Let $\mathfrak{R} = \{b \in \mathfrak{sa}(\mathfrak{B}) : \|b\| \leqslant K\}$ be the closed ball of center 0 and radius K in $\mathfrak{sa}(\mathfrak{B})$. Since $\mathfrak{sa}(\mathfrak{B})$ is finite dimensional, the set $\mathfrak{R}$ is compact in norm. We shall denote by $\mathsf{Haus}[\mathfrak{B}]$ the Hausdorff distance defined by the norm of $\mathfrak{sa}(\mathfrak{B})$ on the compact subsets of $\mathfrak{R}$. Since $\mathfrak{R}$ is compact in norm, $\mathsf{Haus}[\mathfrak{B}]$ induces a compact topology on the set of compact subsets of $\mathfrak{R}$ by Corollary (2.1.12).

We fix a state $\varphi \in \mathscr{S}(\mathfrak{B})$ and identify φ with its restriction to $\mathfrak{A}_{g(n)}$, which is a state of $\mathfrak{A}_{g(n)}$, for all $n \in \mathbb{N}$.

Now, for all $n \in \mathbb{N}$, let:

$$\mathfrak{L}_n = \left\{a \in \mathfrak{sa}\left(\mathfrak{A}_{g(n)}\right) : \mathsf{L}_{g(n)}(a) \leqslant 1\right\}$$

and

$$\mathfrak{D}_n = \left\{a \in \mathfrak{L}_n : \varphi(a) = 0\right\}.$$

Fix $n \in \mathbb{N}$. By construction, we check that $\mathfrak{L}_n = \mathfrak{D}_n + \mathbb{R}1_{\mathfrak{B}}$, since for all $a \in \mathfrak{sa}\left(\mathfrak{A}_{g(n)}\right)$ we check easily that $\mathsf{L}_{g(n)}(a \pm \varphi(a)1_n) = \mathsf{L}_{g(n)}(a)$. On the other other hand, we note that $\mathfrak{D}_n$ is a compact subset of $\mathfrak{R}$ since $\mathrm{diam}\left(\mathscr{S}(\mathfrak{A}_n), \mathsf{mk}_{\mathsf{L}_n}\right) \leqslant K$. Indeed, if $a \in \mathfrak{D}_n$ then, for all $\psi \in \mathscr{S}(\mathfrak{A}_{g(n)})$, we have:

$$|\psi(a)| = |\psi(a) - \varphi(a)| \leqslant \mathsf{mk}_{\mathsf{L}}(\varphi, \psi) \leqslant K.$$

Moreover, compactness of $\mathfrak{D}_n$ follows from Theorem (1.1.19) since $(\mathfrak{A}_{g(n)}, \mathsf{L}_{g(n)})$ is a quantum compact metric space for all $n \in \mathbb{N}$.

Thus, there exists a convergent subsequence $(\mathfrak{D}_{f(n)})_{n\in\mathbb{N}}$ of $(\mathfrak{D}_n)_{n\in\mathbb{N}}$ for $\mathsf{Haus}[\mathfrak{B}]$, whose limit we denote by $\mathfrak{D}$.

We now define $\mathfrak{L} = \mathfrak{D} + \mathbb{R}1_{\mathfrak{B}}$. Let us first check that $(\mathfrak{L}_{f(n)})_{n\in\mathbb{N}}$ converges to $\mathfrak{L}$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have:

$$\mathsf{Haus}[\mathfrak{B}](\mathfrak{D}_{f(n)}, \mathfrak{D}) \leqslant \varepsilon.$$

Let $n \geqslant N$. We observe that, for any $a \in \mathfrak{L}_{f(n)}$, there exists $a' \in \mathfrak{D}_{f(n)}$ and $t \in \mathbb{R}$ such that $a = a' + t1_{\mathfrak{B}}$. Now, there exists $b' \in \mathfrak{D}$ such that $\|a' - b'\|_{\mathfrak{B}} \leqslant \varepsilon$. Let $b = b' + t1_{\mathfrak{B}} \in \mathfrak{L}$. Then $\|a - b\|_{\mathfrak{B}} = \|a' - b'\|_{\mathfrak{B}} \leqslant \varepsilon$, so $\mathfrak{L}_{f(n)}$ is included in an ε-neighborhood of $\mathfrak{L}$. Using a symmetric argument, we conclude:

$$\mathsf{Haus}[\mathfrak{B}]\left(\mathfrak{L}_{f(n)}, \mathfrak{L}\right) \leqslant \varepsilon$$

and thus $\left(\mathfrak{L}_{f(n)}\right)_{n\in\mathbb{N}}$ converges to $\mathfrak{L}$ for the Hausdorff distance $\mathsf{Haus}[\mathfrak{B}]$.

Moreover, $\mathfrak{D} = \{a \in \mathfrak{L} : \varphi(a) = 0\}$ by construction and continuity of φ. Last, as $\mathfrak{D}$ is compact, hence closed, the set $\mathfrak{L} = \mathfrak{D} + \mathbb{R}1_{\mathfrak{B}}$ is closed as well: if $(l_n)_{n\in\mathbb{N}} \in \mathfrak{L}$ converges to some l in $\mathfrak{B}$ then $(l_n - \varphi(l_n)1_{\mathfrak{B}})_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{D}$, and thus by continuity of φ and since $\mathfrak{D}$ is closed, $l - \varphi(l)1_{\mathfrak{B}} \in \mathfrak{D}$. Thus $l \in \mathfrak{L}$.

For all $b \in \mathfrak{sa}(\mathfrak{B})$ we define:

$$\mathsf{L}(b) = \inf\{\lambda > 0 : b \in \lambda\mathfrak{L}\}.$$

A direct computation proves that L is K-quasi-Leibniz. Certainly L may assume the value ∞. Let $\mathfrak{J} = \mathrm{dom}(\mathsf{L})$ be the set of self-adjoint elements in $\mathfrak{B}$ for which L is finite.

If $a, b \in \mathfrak{J}$ then:

$$\mathsf{L}(a \circ b), \mathsf{L}(\{a,b\}) \leqslant F(\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{A}}, \mathsf{L}(a), \mathsf{L}(b)) < \infty$$

so $\mathfrak{J}$ is a Jordan-Lie subalgebra of $\mathfrak{sa}(\mathfrak{B})$. We define:

$$\mathfrak{A} = \{b \in \mathfrak{B} : \Re(b), \Im(b) \in \mathfrak{J}\}$$

and we check that $\mathfrak{A}$ is a C*-subalgebra of $\mathfrak{B}$ with the same unit as $\mathfrak{B}$ and such that $\mathfrak{sa}(\mathfrak{A}) = \mathfrak{J}$.

If $\mathsf{L}(a) = 0$ for some $a \in \mathfrak{J}$, then we have $\mathsf{L}(a - \varphi(a)1_{\mathfrak{A}}) = 0$ as well since $\mathsf{L}(1_{\mathfrak{A}}) = 0$ and L is a seminorm by construction. Thus $a - \varphi(a)1_{\mathfrak{A}} \in \mathfrak{D}$. Now, for any $t \in \mathbb{R}$, we have $\varphi(t(a - \varphi(a)1_{\mathfrak{A}})) = 0$ and $\mathsf{L}(t(a - \varphi(a)1_{\mathfrak{A}})) = 0$, so $t(a - \varphi(a)1_{\mathfrak{A}}) \in \mathfrak{D}$ for all $t \in \mathbb{R}$. Since $\mathfrak{D}$ is norm bounded, we conclude that $a = \varphi(a)1_{\mathfrak{A}}$ as desired.

Since $\mathfrak{D} \subseteq \mathfrak{R}$, for any two states $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ and for all $a \in \mathfrak{L}$ we have $|\varphi(a) - \psi(a)| \leqslant K$ and thus $\mathrm{diam}\left(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}}\right) \leqslant K$.

Moreover, since $\mathfrak{D}$ is compact, we conclude that L is a Leibniz Lip-norm and $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space by Theorem (1.1.19).

We now prove that $\dim \mathfrak{A} \leqslant d$. First, for all $n \in \mathbb{N}$, let $d_n = \dim_{\mathbb{R}} \mathfrak{sa}\left(\mathfrak{A}_{g(f(n))}\right)$. By assumption, since $\dim_{\mathbb{R}} \mathfrak{sa}\left(\mathfrak{A}_{g(f(n))}\right) = \dim_{\mathbb{C}} \mathfrak{A}_{g(f(n))}$, we conclude that $d_n \in \{1, \ldots, d\}$. Thus, there exists a constant subsequence $(d_{f_1(n)})_{n\in\mathbb{N}}$ of $(d_n)_{n\in\mathbb{N}}$. Set $g_1 = g \circ f \circ f_1$ and $\delta = d_{f_1(0)} = d_{f_1(1)} = \ldots$.

Now, for all $n \in \mathbb{N}$, there exists a basis $(c_1^n, \ldots, c_\delta^n)$ of $\mathfrak{sa}\left(\mathfrak{A}_{g_1(n)}\right)$ such that $c_1^n = 1_{\mathfrak{B}}$. In particular, $\mathsf{L}_{g_1(n)}(c_j^n) \in (0, \infty)$ for all $j \in \{2, \ldots, \delta\}$ since $\mathsf{L}_{g_1(n)}$ is a Lip-norm on a finite dimensional space.

Now we set $d_1^n = 1_{\mathfrak{B}}$ and $d_j^n = \mathsf{L}_{g_1(n)}(c_j^n)^{-1} d_j^n$, then we have constructed a basis $(d_1^n, \ldots, d_\delta^n)$ of $\mathfrak{sa}\left(\mathfrak{A}_{g_1(n)}\right)$ consisting of elements in $\mathfrak{L}_{g_1(n)}$. We can improve somewhat on this construction. Indeed, for $j \in \{2, \ldots, \delta\}$, we have $\varphi(d_j^n - \varphi(d_j^n)d_1^n) = 0$. Thus, if $b_1^n = d_1^n$ and $b_j^n = d_j^n - \varphi(d_j^n)d_1^n$, then we have constructed a basis $\left\{b_1^n, \ldots, b_\delta^n\right\}$ of $\mathfrak{A}_{g_1(n)}$ with $b_1^n = 1_{\mathfrak{B}}$ and $b_j^n \in \mathfrak{D}_{g_1(n)}$.

Since $\mathfrak{R}$ is compact and, for any j in the *finite* set $\{1, \ldots, \delta\}$, the sequence $(b_j^n)_{n\in\mathbb{N}}$ lies in $\mathfrak{R}$, there exists a strictly increasing function $f_2 : \mathbb{N} \to \mathbb{N}$ such that for all $j \in \{1, \ldots, \delta\}$, the sequence $(b_j^{f_2(n)})_{n\in\mathbb{N}}$ converges in norm to some $b_j \in \mathfrak{R}$. Let $g_2 = g_1 \circ f_2$.

Since $(\mathfrak{D}_n)_{n\in\mathbb{N}}$ converges to $\mathfrak{D}$ for the Hausdorff distance on $\mathfrak{R}$ associated with the norm of $\mathfrak{B}$, we conclude that $b_j \in \mathfrak{D}$ for all $j \in \{2,\ldots,\delta\}$. Of course, $b_1 = 1_{\mathfrak{A}}$.

Now, let $b \in \mathfrak{sa}(\mathfrak{A})$ be arbitrary. By construction, $a = rb$ for some $r \in \mathbb{R}$ and $a \in \mathfrak{L}$. Since $\mathfrak{L}_{g_2(n)}$ converges to $\mathfrak{L}$, there exists $a_n \in \mathfrak{L}_{g_2(n)}$ such that $\lim_{n\to\infty} a_n = a$. For each $n \in \mathbb{N}$ we write $a_n = \sum_{j=1}^{\delta} \lambda_j^n b_j^n$.

There exists $N_1 \in \mathbb{N}$ such that for all $n \geqslant N_1$, we have $\|a_n\|_{\mathfrak{B}} \leqslant \|a\|_{\mathfrak{B}} + 1$. Thus $\left(\lambda_j^n\right)_{n\in\mathbb{N}}$ is a bounded sequence for all j in the finite set $\{1,\ldots,\delta\}$. Consequently, there exists a strictly increasing $f_3 : \mathbb{N} \to \mathbb{N}$ with $(\lambda_j^{f_3(n)})_{n\in\mathbb{N}}$ converging to some limit $\lambda_j \in \mathbb{R}$ for all $j \in \{1,\ldots,\delta\}$.

Let $\varepsilon > 0$. Summarizing our construction thus far, there exists $N_2 \in \mathbb{N}$ such that for all $n \geqslant N_2$ we have $\|a - a_{f_3(n)}\| \leqslant \frac{\varepsilon}{3}$. There exists $N_3 \in \mathbb{N}$ such that for all $n \geqslant N_3$ and all $j \in \{1,\ldots,\delta\}$ we have $|\lambda_j^{f_3(n)} - \lambda_j| \leqslant \frac{\varepsilon}{3\max\{\|b_1\|_{\mathfrak{B}},\ldots,\|b_\delta\|_{\mathfrak{B}}\}}$. Last there exists $N_4 \in \mathbb{N}$ such that for all $n \geqslant N_4$ we have $\|b_j^{f_2(f_3(n))} - b_j\|_{\mathfrak{B}} \leqslant \frac{\varepsilon}{3\delta\max\{\lambda_1,\ldots,\lambda_\delta,1\}}$.

Now for $n \geqslant \max\{N_2, N_3, N_4\}$:

$$\begin{aligned}
\left\| a - \sum_{j=1}^{\delta} \lambda_j b_j \right\|_{\mathfrak{B}} &\leqslant \|a - a_{f_3(n)}\|_{\mathfrak{B}} + \left\| a_{f_3(n)} - \sum_{j=1}^{\delta} \lambda_j b_j \right\|_{\mathfrak{B}} \\
&\leqslant \frac{\varepsilon}{3} + \left\| a_{f_3(n)} - \sum_{j=1}^{\delta} \lambda_j b_j^{f_2(f_3(n))} \right\|_{\mathfrak{B}} + \left\| \sum_{j=1}^{\delta} \lambda_j \left(b_j^{f_2(f_3(n))} - b_j \right) \right\|_{\mathfrak{B}} \\
&\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Thus a lies in the closure of the span of $\{b_1,\ldots,b_\delta\}$, which is closed since finite dimensional. Hence $\{b_1,\ldots,b_\delta\}$ spans $\mathfrak{A}$ and thus $\dim\mathfrak{A} \leqslant \delta \leqslant d$.

Now, we wish to conclude by showing that $(\mathfrak{A}_{g(f(n))}, \mathsf{L}_{g(f(n))})_{n\in\mathbb{N}}$ converges to $(\mathfrak{A}, \mathsf{L})$ for the quantum propinquity. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have $\mathsf{Haus}[\mathfrak{B}](\mathfrak{L}_{g\circ f(n)}, \mathfrak{L}) \leqslant \varepsilon$. Let now $n \geqslant N$.

For all $a \in \mathfrak{A}_{g\circ f(n)}$ and $b \in \mathfrak{A}$, we set:

$$N_n(a,b) = \frac{1}{\varepsilon}\|a - b\|_{\mathfrak{B}},$$

and

$$\mathsf{L}^n(a,b) = \max\left\{\mathsf{L}_{g\circ f(n)}(a), \mathsf{L}(b), N_n(a,b)\right\}.$$

It is easily checked that N_n is a bridge in the sense of [54, Definition 5.1]: in particular, if $a \in \mathfrak{A}_{g\circ f(n)}$ with $\mathsf{L}_{g\circ f(n)}(a) \leqslant 1$ then there exists $b \in \mathfrak{L}$ with $\|a - b\|_{\mathfrak{B}} \leqslant \varepsilon$, which implies $\mathsf{L}^n(a,b) = 1$; similarly if $b \in \mathfrak{A}$ with $\mathsf{L}(b) \leqslant 1$, i.e. $b \in \mathfrak{L}$, then there exists $a \in \mathfrak{L}_{f(n)}$ with $\|a - b\|_{\mathfrak{B}} \leqslant \varepsilon$ and thus $\mathsf{L}^n(a,b) = 1$.

Hence by [54, Theorem 5.2], the seminorm L^n is a Lip-norm. It is lower semi-continuous by construction, and it is easily checked that L^n is F-quasi-Leibniz, as in our proof of Theorem (2.3.1).

Let $\tau_n = (\mathfrak{A}_{g\circ f(n)} \oplus \mathfrak{A}, \mathsf{L}, \rho_n, \rho)$ with $\rho_n : \mathfrak{A}_{g\circ f(n)} \oplus \mathfrak{A} \twoheadrightarrow \mathfrak{A}_{g\circ f(n)}$ and $\rho : \mathfrak{A}_{g\circ f(n)} \oplus \mathfrak{A} \twoheadrightarrow \mathfrak{A}$ the two canonical surjections. By construction, τ_n is a (C,D)-tunnel.

If $\mu \in \mathscr{S}(\mathfrak{A}_{g\circ f(n)})$ and $\nu \in \mathscr{S}(\mathfrak{A})$ and if $(a,b) \in \mathfrak{A}_{g\circ f(n)} \oplus \mathfrak{A}$ with $\mathsf{L}^n(a,b) \leqslant 1$, then:

$$|\mu(a) - \nu(b)| \leqslant \|a-b\|_{\mathfrak{B}} \leqslant \varepsilon.$$

We can deduce from this computation that $\chi(\tau_n) \leqslant 2\varepsilon$.

Consequently: for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have:

$$\Lambda_F((\mathfrak{A}_{g\circ f(n)}, \mathsf{L}_{g\circ f(n)}), (\mathfrak{A}, \mathsf{L})) \leqslant 2\varepsilon.$$

Moreover, $(\mathfrak{A}, \mathsf{L})$ is a F–quasi-Leibniz quantum compact metric space of diameter at most K and dimension at most d. This completes our proof. □

Our generalization of Gromov's Compactness Theorem [20, 19] is given by:

Theorem 2.5.4. *Let $K \geqslant 1$ and let $\mathscr{A}$ be a class of K-quantum compact metric spaces in the closure of the finite dimensional K-quantum compact metric spaces for the dual propinquity Λ_K. The following assertions are equivalent:*

1. *$\mathscr{A}$ is totally bounded for Λ_K,*
2. *there exists a function $f : (0,\infty) \to \mathbb{N}$ and $C > 0$ such that for all $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \in \mathscr{A}$, we have:*
   - $\operatorname{qdiam}(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \leqslant C$,
   - *for all $\varepsilon > 0$ we have* $\operatorname{cov}_K(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}|\varepsilon) \leqslant f(\varepsilon)$.

Proof. Assume (1). Assertion (2) follows from Proposition (2.5.2).

Assume (2), i.e. assume that there exists $f : (0,\infty) \to \mathbb{N}$ and $K > 0$ such that for all $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \in \mathscr{A}$ and $\varepsilon > 0$, we have $\operatorname{diam}\left(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}}\right) \leqslant K$ and:

$$\operatorname{cov}_F(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}|\varepsilon) \leqslant f(\varepsilon).$$

Let $\varepsilon > 0$.

First, we note that if $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \in \mathscr{A}$, then there exists a F–quasi-Leibniz quantum compact metric space $(\mathfrak{a}, \mathsf{L}_{\mathfrak{a}})$ such that:

- $\dim_{\mathbb{C}} \mathfrak{a} \leqslant f\left(\frac{\varepsilon}{3}\right)$,
- $\Lambda_F((\mathfrak{a}, \mathsf{L}_{\mathfrak{a}}), (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})) \leqslant \frac{\varepsilon}{3}$.

Consequently, we note that $\operatorname{diam}(\mathfrak{a}, \mathsf{L}_{\mathfrak{a}}) \leqslant K + \frac{2\varepsilon}{3}$, since the function $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathscr{A} \mapsto \operatorname{diam}(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is 2-Lipschitz for the Gromov-Hausdorff propinquity by Theorem (2.4.10).

Now, by Theorem (2.5.3), the class:

$$\mathscr{F}_\varepsilon = \left\{ (\mathfrak{B}, \mathsf{L}) \in \mathscr{Q}\mathscr{Q}\mathscr{C}\mathscr{M}\mathscr{S}_F : \dim_{\mathbb{C}} \mathfrak{B} \leqslant f\left(\frac{\varepsilon}{3}\right) \text{ and } \operatorname{diam}(\mathfrak{B}, \mathsf{L}) \leqslant K + \frac{2\varepsilon}{3} \right\}$$

is compact for Λ_F.

Let:

$$\mathscr{G}_\varepsilon = \left\{ (\mathfrak{B}, \mathsf{L}_\mathfrak{B}) \in \mathscr{F}_\varepsilon : \exists (\mathfrak{A}, \mathsf{L}_\mathfrak{A}) \in \mathscr{A} \quad \Lambda_F((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{B}, \mathsf{L}_\mathfrak{B})) \leqslant \frac{\varepsilon}{3} \right\}.$$

Since $\mathscr{G}_\varepsilon \subseteq \mathscr{F}_\varepsilon$, we conclude that $\mathscr{G}_\varepsilon$ is totally bounded for the dual propinquity. Thus, there exists a finite subset $\mathscr{J}_\varepsilon$ of $\mathscr{G}_\varepsilon$ which is $\frac{\varepsilon}{3}$ dense in $\mathscr{G}_\varepsilon$.

Therefore, up to invoking choice, there exists a finite subset $\mathscr{A}_\varepsilon$ of $\mathscr{A}$ such that for all $(\mathfrak{B}, \mathsf{L}_\mathfrak{B}) \in \mathscr{J}_\varepsilon$ there exists $(\mathfrak{A}, \mathsf{L}_\mathfrak{A}) \in \mathscr{A}_\varepsilon$ such that $\Lambda_F((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{B}, \mathsf{L}_\mathfrak{B})) \leqslant \frac{\varepsilon}{3}$.

Now, let $(\mathfrak{A}, \mathsf{L}_\mathfrak{A}) \in \mathscr{A}$. There exists $(\mathfrak{B}, \mathsf{L}_\mathfrak{B}) \in \mathscr{G}_\varepsilon$ such that:

$$\Lambda_F((\mathfrak{B}, \mathsf{L}_\mathfrak{B}), (\mathfrak{A}, \mathsf{L}_\mathfrak{A})) \leqslant \frac{\varepsilon}{3}.$$

Now, there exists $(\mathfrak{C}, \mathsf{L}_\mathfrak{C}) \in \mathscr{J}_\varepsilon$ such that:

$$\Lambda_F((\mathfrak{B}, \mathsf{L}_\mathfrak{B}), (\mathfrak{C}, \mathsf{L}_\mathfrak{C})) \leqslant \frac{\varepsilon}{3}.$$

Last, by our choice, there exists $(\mathfrak{a}, \mathsf{L}_\mathfrak{a}) \in \mathscr{A}_\varepsilon$ with:

$$\Lambda_F((\mathfrak{a}, \mathsf{L}_\mathfrak{a}), (\mathfrak{C}, \mathsf{L}_\mathfrak{C})) \leqslant \frac{\varepsilon}{3}.$$

Consequently:

$$\begin{aligned}
&\Lambda_F((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{a}, \mathsf{L}_\mathfrak{a})) \\
&\quad \leqslant \Lambda_F((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{B}, \mathsf{L}_\mathfrak{B})) + \Lambda_F((\mathfrak{B}, \mathsf{L}_\mathfrak{B}), (\mathfrak{C}, \mathsf{L}_\mathfrak{C})) + \Lambda_F((\mathfrak{C}, \mathsf{L}_\mathfrak{C}), (\mathfrak{a}, \mathsf{L}_\mathfrak{a})) \\
&\quad \leqslant \varepsilon.
\end{aligned}$$

Thus $\mathscr{A}_\varepsilon$ is ε-dense in $\mathscr{A}$ for the dual propinquity, and is a finite set. Thus, $\mathscr{A}$ is totally bounded for Λ_F.

This completes our proof. □

We refer to [37] for the following example of compact classes.

Theorem 2.5.5. *Let $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ be a F–quasi-Leibniz quantum compact metric space for some admissible function F. If $R \geqslant 0$ then the class $\mathscr{B}$ of F–quasi-Leibniz quantum compact metric spaces in the closed ball of center $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ and radius R for the Lipschitz distance LipD is totally bounded for the quantum Gromov-Hausdorff propinquity.*

Therefore, the closure of $\mathscr{B}$ for the dual propinquity is compact.

Other compact classes are obtained via various form of perturbation, even when the space of deformation parameters is very far from compact.

2.6 TOPOLOGICAL EQUIVALENCE

A very important of our compactness result is that our metric induces the same topology as the usual Gromov-Hausdorff distance on the class of classical compact metric spaces.

If (X,d) is a compact metric space, then we write L_d for the Lipschitz seminorm induced by d, i.e.

$$\forall f\in C(X) \quad \mathsf{L}(f) := \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} : x,y\in X, x\neq y\right\},$$

allowing for the value ∞.

Let F be the class function which, to any compact metric space (X,d), associates the quantum compact metric space $(C(X),\mathsf{L}_d)$. We can turn F into a functor, by setting $F(f) : g\in C(Y)\mapsto f\circ g$ for any isometry $g:(X,d_X)\to(Y,d_Y)$, where then $F(f)$ is a quantum isometry.

If $F(X,d_X)$ and $F(Y,d_Y)$ are fully quantum isometric, then (X,d_X) and (Y,d_Y) are in fact, fully isometric. Indeed, a full quantum isometry from $(C(X),d_X)$ onto $(C(Y),d_Y)$ gives rise, as a *-isomorphism from $C(X)$ to $C(Y)$, to an homeomorphism $f:Y\to X$. Moreover, by definition of a full quantum isometry, $\mathsf{L}_{d_Y}(g\circ f)=\mathsf{L}_{d_X}(g)$ for all $g\in C(X)$ (again, we allow ∞ here) and $\mathsf{L}_{d_X}(g\circ f^{-1})=\mathsf{L}_{d_Y}(g)$ for all $g\in C(Y)$. In particular, $\{g\circ f:\mathsf{L}_{d_X}(g)\leqslant 1\}=\{h\in C(Y):\mathsf{L}_{d_Y}(h)\leqslant 1\}$. Therefore, for all $x,y\in X$, we compute:

$$\begin{aligned} d_X(x,y) &= \sup\left\{|g(x)-g(y)| : g\in\mathfrak{sa}(C(X)), \mathsf{L}_{d_X}(g)\leqslant 1\right\}\\ &= \sup\left\{|g\circ f(f^{-1}(x))-g\circ f(f^{-1}(b))| : g\in\mathfrak{sa}(C(X)), \mathsf{L}_{d_X}(g)\leqslant 1\right\}\\ &= \sup\left\{|h(f^{-1}(x))-h(f^{-1}(y)) : h\in sa\mathbb{C}(Y), \mathsf{L}_{d_Y}(h)\leqslant 1\right\}\\ &= d_Y(f^{-1}(x),f^{-1}(y)). \end{aligned}$$

So f^{-1} is an isometry from (X,d_X) to (Y,d_Y). Similarly, f is an isometry and thus (X,d_X) and (Y,d_Y) are fully isometric.

We first observe that F is 1-Lipschitz.

Proposition 2.6.1. *For all compact metric spaces (X,d_X) and (Y,d_Y),*

$$\Lambda^*(F(X,d_x),F(Y,d_Y))\leqslant \mathsf{GH}((X,d_X),(Y,d_Y)).$$

Proof. Let (Z,d_Z) be any compact metric space, together with two isometries $i_X : X\hookrightarrow Z$ and $i_Y:Y\hookrightarrow Z$. We thus have a tunnel $\tau : F(X,d_X)\xleftarrow{i_X^*} F(Z,d_Z)\xrightarrow{i_Y^*} F(Y,d_Y)$ from $F(X,d_X)$ to $F(Y,d_Y)$. We write δ_x for the evaluation at x map. Moreover, let $W:=i_X(X)\cup i_Y(Y)\subseteq Z$, endowed with the restriction d_W of d_Z to W. Of course, $\mathsf{Haus}[d_W](i_X(X),i_Y(Y))=\mathsf{Haus}[d_Z](i_X(X),i_Y(Y))$.

Let $\varphi:=\sum_{j=1}^n t_j\delta_{x_j}$ for some $x_1,\ldots,x_n\in W$, and $t_1,\ldots,t_n\in[0,1]$ such that $\sum_{j=1}^n t_j = 1$. Note that $\varphi\in\mathscr{S}(C(W))$. For each $j\in\{1,\ldots,n\}$, if $x_j\in Y$ then set $y_j=x_j$; otherwise, let y_j be chosen so that $d_W(i_X(x_j),i_Y(y_j))\leqslant\mathsf{Haus}[d_W](i_X(X),i_Y(Y))$.

Let $\psi := \sum_{j=1}^n t_j \delta_{y_j} \in \mathscr{S}(C(Y))$. If $f \in F(W, d_W)$, with $\mathsf{L}_{d_W}(f) \leqslant 1$, then

$$\begin{aligned}|\varphi(f) - \psi(f)| &\leqslant \sum_{j=1}^n t_j |f(x_j) - f(y_j)|\\ &\leqslant \sum_{j=1}^n t_j d_W(x_j, y_j)\\ &\leqslant \sum_{j=1}^n t_j \mathsf{Haus}[d_W](i_X(X), i_Y(Y)) = \mathsf{Haus}[d_W](i_X(X), i_Y(Y)).\end{aligned}$$

The set of states obtained as convex combinations of evaluation states at points in W is weak* dense in $\mathscr{S}(C(W))$. Since $\mathsf{mk}_{\mathsf{L}_{d_W}}$ metrizes the weak* topology on $\mathscr{S}(C(W))$, we conclude that $\mathsf{Haus}\left[\mathsf{mk}_{\mathsf{L}_{d_W}}\right](\mathscr{S}(C(W)), \mathscr{S}(C(Y))) \leqslant \mathsf{Haus}[d_W](i_X(X), i_Y(Y))$.

Similarly, $\mathsf{Haus}\left[\mathsf{mk}_{\mathsf{L}_{d_W}}\right](\mathscr{S}(C(W)), \mathscr{S}(C(X))) \leqslant \mathsf{Haus}[d_W](i_X(X), i_Y(Y))$. So $\chi(\tau) \leqslant \mathsf{Haus}[d_W](i_X(X), i_Y(Y))$. So

$$\Lambda^*(F(X, d_X), F(Y, d_Y)) \leqslant \mathsf{Haus}[d_W](i_X(X), i_Y(Y)).$$

By definition of GH, we conclude (as our choices of (Z, d_X) and i_X, i_Y are arbitrary within the given assumptions):

$$\Lambda^*(F(X, d_X), F(Y, d_Y)) \leqslant \mathsf{GH}((X, d_X), (Y, d_Y)),$$

as claimed. □

We now get the topological equivalence between the Gromov-Hausdorff distance and the propinquity on the class of classical compact metric spaces.

Theorem 2.6.2. *The topology defined by the distance:*

$$\Lambda_c : (X, d_X), (Y, d_Y) \text{ compact metric spaces} \mapsto \Lambda^*(F(X, d_X), F(Y, d_Y))$$

is the topology induced by the Gromov-Hausdorff distance GH*, and the class* $\{F(X, d_X) : (X, d_X)$ *compact metric space*$\}$ *is closed for the propinquity.*

Proof. By Proposition (2.6.1), convergence for the Gromov-Hausdorff distance implies convergence for Λ_c.

Let now $(X_n, d_n)_{n\in\mathbb{N}}$ be a sequence of compact metric spaces, and let $(\mathfrak{A}, \mathsf{L})$ be a quantum compact metric space such that

$$\lim_{n\to\infty} \Lambda^*(F(X_n, d_n), (\mathfrak{A}, \mathsf{L})) = 0.$$

Since $F(X_n, d_n)$ converges for the propinquity, it is Cauchy, and thus the set $\{F(X_n, d_n) : n \in \mathbb{N}\}$ is totally bounded. Moreover, every compact metric space is in the closure of finite metric spaces for GH, and thus, for Λ, by Proposition (2.6.1). So by Proposition (2.5.2), there exists $f : (0, \infty) \to \mathbb{N}$ and $K > 0$ such that for all $n \in \mathbb{N}$ and for all $\varepsilon > 0$, we have $\operatorname{diam}(X_n, d_n) \leqslant K$ and $\operatorname{cov}((|F)(X_n, d_n)) \leqslant f(\varepsilon)$.

As a result, $\{(X_n,d_n) : n \in \mathbb{N}\}$ is totally bounded for the Gromov-Hausdorff distance GH, thanks to the Gromov compactness theorem [20].

As a result, we observe the following. Let $g : \mathbb{N} \to \mathbb{N}$ be some strictly increasing function. By total boundedness, the sequence $(X_{g(n)}, d_{g(n)})_{n\in\mathbb{N}}$ has a Cauchy subsequence $(X_{g(h(n))}, \mathsf{L}_{g(h(n))})_{n\in\mathbb{N}}$. Since the Gromov-Hausdorff distance is complete [20], we conclude that there exists a compact metric space (Y_g, d_g) such that

$$\lim_{n\to\infty} \mathsf{GH}((X_{g(h(n))}, d_{g(h(n))}), (Y_g, d_g)) = 0.$$

By Proposition (2.6.1), we then conclude that

$$\lim_{n\to\infty} \Lambda^*((C(X_{g(h(n))}), \mathsf{L}_{g(h(n))}), (C(Y_g), \mathsf{L}_{d_g})) = 0.$$

By Theorem (2.4.3), this implies in turn that $(\mathfrak{A}, \mathsf{L})$ is fully quantum isometric to $(C(Y_g), \mathsf{L}_{d_g})$. In particular, the class of classical compact metric spaces is closed for the propinquity.

Thus, on the class of isometric classes of compact metric spaces, we have shown the following: if $(X_n,d_n)_{n\in\mathbb{N}}$ is a sequence of compact metric spaces such that $(C(X_n), \mathsf{L}_n)_{n\in\mathbb{N}}$ converges for the dual propinquity, then all subsequences of $(X_n,d_n)_{n\in\mathbb{N}}$ have a convergent subsequence for the Gromov-Hausdorff distance GH, and this limit (Y,d) is such that $(C(Y), \mathsf{L}_d)$ is fully quantum isometric to $(\mathfrak{A}, \mathsf{L})$ — in other words, all subsequences of $(X_n,d_n)_{n\in\mathbb{N}}$ have a subsequence converging to the same element in the class $\mathscr{C}$ of all isometry classes of compact metric spaces. As GH is indeed a metric on $\mathscr{C}$, we conclude that $(X_n,d_n)_{n\in\mathbb{N}}$ converges in the Gromov-Hausdorff distance GH, with its limit (Y,d) such that $(\mathfrak{A}, \mathsf{L})$ and $(C(Y), \mathsf{L}_d)$ are fully quantum isometric. This concludes our proof. $\square$

Thus, convergence for the propinquity or for the Gromov-Hausdorff distance are the same thing when dealing with classical compact metric spaces.

2.7 INDUCTIVE LIMITS

In general, it is difficult to find interesting necessary conditions for convergence of quantum compact metric spaces, since the Gromov-Hausdorff propinquity is defined an infimum over all possible tunnels. Of course, Theorem (2.4.10) is such a necessary condition, but it is not always easy to compute the diameter of a quantum compact metric space anyway. However, something notable happens when working with inductive limits of quantum compact metric spaces, in the relatively lax sense below: we can actually find a necessary and sufficient condition for convergence in that context, under a mild assumption.

Definition 2.7.1. For each $n \in \mathbb{N} \cup \{\infty\}$, let $(\mathfrak{A}_n, \mathsf{L}_n)$ be a quantum compact metric space, such that $\mathfrak{A}_\infty = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right)$, where $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ is an increasing (for $\subseteq$) sequence of C*-subalgebras of $\mathfrak{A}_\infty$, with the unit of $\mathfrak{A}_\infty$ in $\mathfrak{A}_0$.

A $*$-automorphism $\pi : \mathfrak{A}_\infty \to \mathfrak{A}_\infty$ is a *bridge builder* for $((\mathfrak{A}_n, \mathsf{L}_n)_{n\in\mathbb{N}}, (\mathfrak{A}_\infty, \mathsf{L}_\infty))$ when, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geqslant N$, then

$$\forall a \in \mathrm{dom}(\mathsf{L}_\infty) \quad \exists b \in \mathrm{dom}(\mathsf{L}_n) : \quad \mathsf{L}_n(b) \leqslant \mathsf{L}_\infty(a) \text{ and } \|\pi(a) - b\|_{\mathfrak{A}_\infty} < \varepsilon \mathsf{L}_\infty(a)$$

and

$$\forall b \in \mathrm{dom}(\mathsf{L}_n) \quad \exists a \in \mathrm{dom}(\mathsf{L}_\infty): \quad \mathsf{L}_\infty(a) \leqslant \mathsf{L}_n(b) \text{ and } \|\pi(a) - b\|_{\mathfrak{A}_\infty} < \varepsilon \mathsf{L}_n(b),$$

where $\|\cdot\|_{\mathfrak{A}_\infty}$ is the C*-norm on $\mathfrak{A}_\infty$.

Bridge builders are powerful means to prove metric convergence for the propinquity and notable because it is usually very difficult to find necessary conditions for metric convergence in the sense of the propinquity (besides the trivial convergence for the diameters). Thus, this theorem is of independent interest from our study of spectral triples, and addresses the relationship between inductive limits and limits in a metric sense as in [47, 36]. Our first main result is therefore the following theorem about convergence for the propinquity Λ^* of certain inductive sequences.

Theorem 2.7.2. *For each $n \in \mathbb{N} \cup \{\infty\}$, let $(\mathfrak{A}_n, \mathsf{L}_n)$ be a quantum compact metric space, where $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ is an increasing (for $\subseteq$) sequence of C*-subalgebras of $\mathfrak{A}_\infty$ such that $\mathfrak{A}_\infty = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right)$, with the unit of $\mathfrak{A}_\infty$ in $\mathfrak{A}_0$. We assume that there exists $\exists M > 0$ such that for all $n \in \mathbb{N}$:*

$$\frac{1}{M}\mathsf{L}_n \leqslant \mathsf{L}_\infty \leqslant M \cdot \mathsf{L}_n \text{ on } \mathrm{dom}(\mathsf{L}_n).$$

Then

$$\lim_{n\to\infty} \Lambda^*\left((\mathfrak{A}_n, \mathsf{L}_n), (\mathfrak{A}_\infty, \mathsf{L}_\infty)\right) = 0,$$

if, and only if, for any subsequence $(\mathfrak{A}_{g(n)}, \mathsf{L}_{g(n)})_{n\in\mathbb{N}}$ of $(\mathfrak{A}_n, \mathsf{L}_n)_{n\in\mathbb{N}}$, there exists a strictly increasing function $f: \mathbb{N} \to \mathbb{N}$ and a bridge builder *π for $((\mathfrak{A}_{g\circ f(n)}, \mathsf{L}_{g\circ f(n)})_{n\in\mathbb{N}}, (\mathfrak{A}_\infty, \mathsf{L}_\infty))$.*

Example 2.7.3. Let $\mathfrak{A} = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}}\right)\mathfrak{A}_n$ be a unital C*-algebra, where $\mathfrak{A}_n$ is a finite dimensional C*-subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$ for all $n \in \mathbb{N}$, and $1 \in \mathfrak{A}_0$ — i.e. $\mathfrak{A}$ is a unital AF algebra. We assume that $\mathfrak{A}$ has a faithful tracial state τ, which is equivalent to asking that $\mathfrak{A}$ does not contain the C*-algebra of compact operators.

For each $n \in \mathbb{N}$, there exists a unique conditional expectation $\mathbb{E}_n: \mathfrak{A} \to \mathfrak{A}_n$ such that $\tau \circ \mathbb{E}_n = \tau$.

Let $(d_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers converging to ∞; for instance, $d_n = \dim \mathfrak{A}_n$ for all $n \in \mathbb{N}$. We define, for all $a \in \mathfrak{sa}(\mathfrak{A})$,

$$\mathsf{L}(a) := \sup\left\{d_n \|a - \mathbb{E}_n(a)\|_{\mathfrak{A}} : n \in \mathbb{N}\right\},$$

allowing for the value ∞.

The pair $(\mathfrak{A}, \mathsf{L})$ thus defined is a quantum compact metric space such that

$$\lim_{n\to\infty} \Lambda^*((\mathfrak{A}_n, \mathsf{L}_n), (\mathfrak{A}, \mathsf{L})) = 0.$$

The identity is a bridge builder, almost immediately from the definition of L.

We refer to [7] for examples involving inductive limits of twisted group C*-algebras (see the last section of these notes as well).

2.8 OTHER EXAMPLES

There are other examples of convergence, and we include a few here. To begin with, every nuclear quasi-diagonal C*=algebra can be endowed with a quantum compact metric space structure, and be the limit, for this structure, of finite dimensional quantum compact metric spaces.

NUCLEAR QUASI-DIAGONAL QUANTUM COMPACT METRIC SPACES

Definition 2.8.1. A unital C*-algebra $\mathfrak{A}$ is *pseudo-diagonal*, when for all $\varepsilon > 0$ and for all finite subset $\mathfrak{F}$ of $\mathfrak{A}$, there exist a finite dimensional C*-algebra $\mathfrak{B}$ and two positive unital linear maps $\varphi : \mathfrak{B} \to \mathfrak{A}$ and $\psi : \mathfrak{A} \to \mathfrak{B}$ such that:

1. for all $a \in \mathfrak{F}$, we have $\|a - \varphi \circ \psi(a)\|_{\mathfrak{A}} \leqslant \varepsilon$,
2. for all $a, b \in \mathfrak{F}$, we have $\|\psi(a \circ b) - \psi(a) \circ \psi(b)\|_{\mathfrak{B}} \leqslant \varepsilon$,
3. for all $a, b \in \mathfrak{F}$, we have $\|\psi(\{a, b\}) - \left\{\psi(a), \psi(b)\right\}\|_{\mathfrak{B}} \leqslant \varepsilon$.

Theorem 2.8.2 ([39])**.** *Let $C \geqslant 1$, $D \geqslant 0$. If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is a (C, D)-quantum compact metric space, where $\mathfrak{A}$ is a unital quasi-diagonal C*-algebra, then $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is the limit of finite dimensional $(C(1+\varepsilon), D+\varepsilon)$-quantum compact metric spaces for the $(C(1+\varepsilon), D+\varepsilon)$-propinquity, for any $\varepsilon > 0$.*

AF ALGEBRAS

We saw how to endow AF algebras with quantum metrics in Theorem (1.2.3). We can use this metric structure to obtain various continuous functions from natural spaces to the class of AF algebras. It is actually possible to get a decent view of UHF algebras in this picture.

Up to unitary conjugation, a unital *-monomorphism $\alpha : \mathfrak{B} \to \mathfrak{A}$ between two unital simple finite dimensional C*-algebras, i.e. two nonzero full matrix algebras $\mathfrak{A}$ and $\mathfrak{B}$, exists if and only if $\dim \mathfrak{A} = k^2 \dim \mathfrak{B}$ for $k \in \mathbb{N}$, and α must be of the form:

$$A \in \mathfrak{B} \longmapsto \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} \in \mathfrak{A}. \tag{2.8.1}$$

It is thus sufficient, in order to characterize a unital inductive sequence of full matrix algebras, to give a sequence of positive integers:

Definition 2.8.3. Let $\mathscr{I} = (\mathfrak{A}_n, \alpha_n)_{n \in \mathbb{N}}$ be a unital inductive sequence of unital, simple finite dimensional C*-algebras, with $\mathfrak{A}_0 = \mathbb{C}$.

The *multiplicity sequence* of $\mathscr{I}$ is the sequence $\left(\sqrt{\frac{\dim \mathfrak{A}_{n+1}}{\dim \mathfrak{A}_n}}\right)_{n \in \mathbb{N}}$ of positive integers.

The set of sequences $\mathscr{N}$ of positive integers is thus a natural parameter space for the classes $\mathscr{UHF}^k$. Moreover, $\mathscr{N}$ can be endowed with a natural topology, and we thus can investigate the continuity of maps from the Baire space to $\left(\mathscr{UHF}^k, \Lambda\right)$.

Definition 2.8.4. The *Baire space* $\mathscr{N}$ is the set $(\mathbb{N}\setminus\{0\})^{\mathbb{N}}$ endowed with the metric d defined, for any two $(x(n))_{n\in\mathbb{N}}$, $(y(n))_{n\in\mathbb{N}}$ in $\mathscr{N}$, by:

$$\mathsf{d}\left((x(n))_{n\in\mathbb{N}},(y(n))_{n\in\mathbb{N}}\right)=\begin{cases}0 \text{ if } x(n)=y(n) \text{ for all } n\in\mathbb{N},\\ 2^{-\min\{n\in\mathbb{N}:x(n)\neq y(n)\}} \text{ otherwise.}\end{cases}$$

We thus can establish:

Theorem 2.8.5 ([2, Theorem 4.9]). *For any $\beta=(\beta(n))_{n\in\mathbb{N}}\in\mathscr{N}$, we define the sequence $\boxtimes\beta$ by:*

$$\boxtimes\beta = n\in\mathbb{N}\longmapsto\begin{cases}1 \text{ if } n=0,\\ \prod_{j=0}^{n-1}(\beta(j)+1) \text{ otherwise.}\end{cases}$$

We then define, for all $\beta\in\mathscr{N}$, the unital inductive sequence:

$$\mathscr{I}(\beta)=\left(\mathfrak{M}\left(\boxtimes\beta(n)\right),\alpha_n\right)_{n\in\mathbb{N}}$$

*where $\mathfrak{M}(d)$ is the algebra of $d\times d$ matrices and for all $n\in\mathbb{N}$, the unital *-monomorphism α_n is of the form given in Expression (2.8.1).*

The map $\mathfrak{u}$ from $\mathscr{N}$ to the class of UHF algebras is now defined by:

$$(\beta(n))_{n\in\mathbb{N}}\in\mathscr{N}\longmapsto\mathfrak{u}((\beta(n))_{n\in\mathbb{N}})=\varinjlim\mathscr{I}(\beta).$$

Let $k\in(0,\infty)$ and $\beta\in\mathscr{N}$. Let L_β^k be the Lip-norm $\mathsf{L}^{\vartheta}_{\mathscr{I}(\beta),\mu}$ on $\mathfrak{u}(\beta)$ given by Theorem (1.2.3), the sequence $\vartheta: n\in\mathbb{N}\mapsto\boxtimes\beta(n)^k$ and the unique faithful trace μ on $\mathfrak{u}(\beta)$.

The $(2,0)$–quasi-Leibniz quantum compact metric space $\left(\mathfrak{u}(\beta),\mathsf{L}_\beta^k\right)$ will be denoted simply by $\mathfrak{uhf}(\beta,k)$.

For all $k\in(0,\infty)$, the map:

$$\mathfrak{uhf}(\cdot,k):\mathscr{N}\longrightarrow\mathscr{UHF}^k$$

is a $(2,k)$-Hölder surjection.

Another family of AF algebras which is also parameterized by the Baire space and played a role in the classification of quantum tori is the family of Effros-Shem algebras [16].

We begin by recalling the construction of the AF C*-algebras $\mathfrak{AF}_\theta$ constructed in [17] for any irrational θ in $(0,1)$. For any $\theta\in(0,1)\setminus\mathbb{Q}$, let $(r_j)_{j\in\mathbb{N}}$ be the unique sequence in $\mathbb{N}$ such that:

$$\theta=\lim_{n\to\infty} r_0+\cfrac{1}{r_1+\cfrac{1}{r_2+\cfrac{1}{r_3+\cfrac{1}{\ddots+\cfrac{1}{r_n}}}}}. \tag{2.8.2}$$

The sequence $(r_j)_{j\in\mathbb{N}}$ is called the continued fraction expansion of θ, and we will simply denote it by writing $\theta = [r_0, r_1, r_2, \ldots] = [r_j]_{j\in\mathbb{N}}$. We note that $r_0 = 0$ (since $\theta \in (0,1)$) and $r_n \in \mathbb{N}\setminus\{0\}$ for $n \geqslant 1$.

We fix $\theta \in (0,1)\setminus\mathbb{Q}$, and let $\theta = [r_j]_{j\in\mathbb{N}}$ be its continued fraction decomposition. We then obtain a sequence $\left(\frac{p_n^\theta}{q_n^\theta}\right)_{n\in\mathbb{N}}$ with $p_n^\theta \in \mathbb{N}$ and $q_n^\theta \in \mathbb{N}\setminus\{0\}$ by setting:

$$\begin{cases} \begin{pmatrix} p_1^\theta & q_1^\theta \\ p_0^\theta & q_0^\theta \end{pmatrix} = \begin{pmatrix} r_0 r_1 + 1 & r_1 \\ r_0 & 1 \end{pmatrix} \\ \begin{pmatrix} p_{n+1}^\theta & q_{n+1}^\theta \\ p_n^\theta & q_n^\theta \end{pmatrix} = \begin{pmatrix} r_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n^\theta & q_n^\theta \\ p_{n-1}^\theta & q_{n-1}^\theta \end{pmatrix} \text{ for all } n \in \mathbb{N}\setminus\{0\}. \end{cases} \tag{2.8.3}$$

We then note that $\left(\frac{p_n^\theta}{q_n^\theta}\right)_{n\in\mathbb{N}}$ converges to θ.

Expression (2.8.3) contains the crux for the construction of the Effrös-Shen AF algebras.

Notation 2.8.6. Let $\theta \in (0,1)\setminus\mathbb{Q}$ and $\theta = [r_j]_{j\in\mathbb{N}}$ be the continued fraction expansion of θ. Let $(p_n^\theta)_{n\in\mathbb{N}}$ and $(q_n^\theta)_{n\in\mathbb{N}}$ be defined by Expression (2.8.3). We set $\mathfrak{AF}_{\theta,0} = \mathbb{C}$ and, for all $n \in \mathbb{N}\setminus\{0\}$, we set:

$$\mathfrak{AF}_{\theta,n} = \mathfrak{M}(q_n^\theta) \oplus \mathfrak{M}(q_{n-1}^\theta),$$

and:

$$\alpha_{\theta,n} : a \oplus b \in \mathfrak{AF}_{\theta,n} \longmapsto \begin{pmatrix} a & & & \\ & \ddots & & \\ & & a & \\ & & & b \end{pmatrix} \oplus a \in \mathfrak{AF}_{\theta,n+1},$$

where a appears r_{n+1} times on the diagonal of the right hand side matrix above. We also set α_0 to be the unique unital *-morphism from $\mathbb{C}$ to $\mathfrak{AF}_{\theta,1}$.

We thus define the Effrös-Shen C*-algebra $\mathfrak{AF}_\theta$, after [17]:

$$\mathfrak{AF}_\theta = \varinjlim \left(\mathfrak{AF}_{\theta,n}, \alpha_{\theta,n}\right)_{n\in\mathbb{N}}.$$

Notation 2.8.7. Let $\theta \in (0,1)\setminus\mathbb{Q}$ and $\theta = [r_j]_{j\in\mathbb{N}}$ be the continued fraction expansion of θ. Let $(p_n^\theta)_{n\in\mathbb{N}}$ and $(q_n^\theta)_{n\in\mathbb{N}}$ be defined by Expression (2.8.3). We set $\mathfrak{af}(\theta,0) = \mathbb{C}$ and, for all $n \in \mathbb{N}\setminus\{0\}$, we set:

$$\mathfrak{af}(\theta,n) = \mathfrak{M}(q_n^\theta) \oplus \mathfrak{M}(q_{n-1}^\theta),$$

and:

$$\alpha_{\theta,n} : a \oplus b \in \mathfrak{af}(\theta,n) \longmapsto \begin{pmatrix} a & & & \\ & \ddots & & \\ & & a & \\ & & & b \end{pmatrix} \oplus a \in \mathfrak{AF}_{\theta,n+1},$$

where a appears r_{n+1} times on the diagonal of the right hand side matrix above. We also set α_0 to be the unique unital *-morphism from $\mathbb{C}$ to $\mathfrak{AF}_{\theta,1}$.

We thus define the Effrös-Shen C*-algebra $\mathfrak{AF}_\theta$, after [17]:

$$\mathfrak{AF}_\theta = \varinjlim \left(\mathfrak{AF}_{\theta,n}, \alpha_{\theta,n}\right)_{n\in\mathbb{N}}.$$

Notation 2.8.8. Let $\theta \in (0,1)\setminus\mathbb{Q}$ and $k \in (0,\infty)$. The Lip-norm L_θ^k on $\mathfrak{AF}_\theta$ is the lower semi-continuous, $(2,0)$-quasi Leibniz Lip-norm $\mathsf{L}^k_{\mathscr{I}(\theta),\sigma_\theta}$ defined in Theorem (1.2.3), where $\mathscr{I}(\theta) = (\mathfrak{AF}_{\theta,n}, \alpha_{\theta,n})_{n\in\mathbb{N}}$ as in Notation (2.8.7).

+

Theorem 2.8.9. *For all $k \in (0,\infty)$ and using Notations (2.8.7) and (2.8.8), the function:*

$$\theta \in (0,1)\setminus\mathbb{Q} \longmapsto \left(\mathfrak{AF}_\theta, \mathsf{L}_\theta^k\right) \in \mathscr{AF}^k$$

is continuous from $(0,1)\setminus\mathbb{Q}$, with its topology as a subset of $\mathbb{R}$, to the class of $(2,0)$-quasi-Leibniz quantum compact metric spaces metrized by the propinquity Λ.

QUANTUM AND FUZZY TORI

Let ℓ be a *continuous length function* on $\mathbb{T}^d$. For any $G \subseteq \mathbb{T}^d$ a closed subgroup and σ a multiplier of the Pontryagin dual $\widehat{G}$ of G, for any $a \in C^*(\widehat{G},\sigma)$, set:

$$\mathsf{L}_{G,\sigma}(a) = \sup\left\{ \frac{\|\alpha^g(a) - a\|_{C^*(\widehat{G},\sigma)}}{\ell(g)} : g \in G\setminus\{1\}\right\}$$

where α is the dual action of G on $C^*(\widehat{G},\sigma)$.

Rieffel showed in [52] that $\left(C^*(\widehat{G},\sigma), \mathsf{L}_{G,\sigma}\right)$ is a Leibniz quantum compact metric space.

If $(G_n)_{n\in\mathbb{N}}$ is a sequence of closed subgroups of $\mathbb{T}^d$ converging to $\mathbb{T}^d$ for the Hausdorff distance $\mathsf{Haus}[\ell]$ induced by the length function ℓ on the hyperspace of closed subsets of $\mathbb{T}^d$, and if $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of multipliers of $\mathbb{Z}^d$ converging pointwise to some σ, with $\sigma_n(g) = 1$ if g is the coset of 0 for $\widehat{G_n}$, then:

$$\lim_{n\to\infty} \Lambda^*((C^*(\widehat{G_n},\sigma_n), \mathsf{L}_{\widehat{G_n},\sigma_n}), (C^*(\mathbb{Z}^d,\sigma), \mathsf{L}_{\mathbb{Z}^d,\sigma})) = 0.$$

In particular, we obtain certain finite dimensional approximations of quantum tori. If for all $n \in \mathbb{N}$, we set $\mathscr{F}_n = C^*(U_n, V_n) = C^*(\mathbb{Z}_n^2, \rho_n)$ where:

$$U_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & \dots & 1 & 0 \end{pmatrix}, V_n = \begin{pmatrix} 1 & & & & \\ & e^{\frac{2ip_n\pi}{n}} & & & \\ & & e^{\frac{4ip_n\pi}{n}} & & \\ & & & \ddots & \\ & & & & e^{\frac{2i(n-1)p_n\pi}{n}} \end{pmatrix}$$

with $p_n \not\equiv 0 \mod n$, and if $\lim_{n\to\infty} \frac{p_n}{n} = \theta$, then:

$$\lim_{n\to\infty} \Lambda^*((\mathscr{F}_n, \mathsf{L}_n), (\mathscr{A}_\theta, \mathsf{L}_\theta)) = 0$$

where $\mathscr{A}_\theta = C^*(U,V)$ and U,V are universal unitaries such that $VU = e^{2i\pi\theta}UV$, while L_n and L are L-seminorms from the dual actions, and for some fixed continuous length function on $\mathbb{T}^2$.

THE TOPOLOGY OF THE CLASS OF CLASSICAL COMPACT METRIC SPACES

In this section, we note that, if we are willing to work with $(2,0)$-quantum compact metric spaces and the associated propinquity, then we can say a little bit more about the topology of the class of compact metric spaces.

Lemma 2.8.10. *If $\mathfrak{B}$ is a finite dimensional C*-subalgebra of a unital C*-algebra $\mathfrak{A}$ and $1_{\mathfrak{A}} \in \mathfrak{B}$ and if $\mathfrak{A}$ has a faithful tracial state $\mu \in \mathscr{S}(\mathfrak{A})$ then there exists a unique μ-preserving conditional expectation $\mathbb{E} : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$.*

Proof. See [35, Step 1 of Theorem (3.5)]. □

Theorem 2.8.11. *Let (X,d) be a finite metric space and let:*

$$\delta = \min\left\{d(x,y) : x,y \in X, x \neq y\right\} > 0.$$

If $\mathfrak{A}$ is a finite dimensional C-algebra, if τ is some faithful tracial state on $\mathfrak{A}$, and if $\mathfrak{B}$ is a C*-subalgebra of $\mathfrak{A}$ such that:*

1. $1_{\mathfrak{A}} \in \mathfrak{B}$,
2. *there exists a unital *-isomorphism $\rho : C(X) \rightarrow \mathfrak{B}$,*

then, for any $\beta > 0$, and setting for all $a \in \mathfrak{A}$:

$$\mathsf{L}(a) = \max\left\{\frac{\|a - \mathbb{E}(a)\|_{\mathfrak{A}}}{\beta}, \mathsf{Lip}_d \circ \rho^{-1}(\mathbb{E}(a)), \right\}$$

where $\mathbb{E} : \mathfrak{A} \rightarrow \mathfrak{B}$ is the conditional expectation such that $\tau \circ \mathbb{E} = \tau$, we conclude that the space $(\mathfrak{A}, \mathsf{L})$ is a $(D,0)$-quasi-Leibniz compact quantum metric space, where:

$$D = \max\left\{2, 1 + \frac{\beta}{\delta}\right\}$$

such that:

$$\Lambda\left((\mathfrak{A},\mathsf{L}),(C(X),\mathsf{Lip}_d)\right) \leqslant \beta.$$

Proof. If $a \in \mathfrak{A}$ with $\mathsf{L}(a) = 0$ then $a = \mathbb{E}(a)$, and $\mathsf{Lip}_d(\rho^{-1}(\mathbb{E}(a))) = 0$, so $\mathbb{E}(a) = \lambda 1_{\mathfrak{A}}$ for some $\lambda \in \mathbb{R}$. Thus $a \in \mathbb{R}1_{\mathfrak{A}}$, as desired. We also note that $\mathsf{L}(1_{\mathfrak{A}}) = 0$ by assumption.

We also note that since X is finite, $\operatorname{dom}\left(\mathsf{Lip}_d\right) = C(X)$ so $\operatorname{dom}(\mathsf{L}) = \mathfrak{A}$.

Since L is the maximum of two (lower semi-)continuous functions over $\mathfrak{A}$, we also have L is (lower semi-)continuous on $\mathfrak{A}$.

The map $\tau_X = \tau \circ \rho$ is a state of $C(X)$, and thus $\{f \in C(X) : \tau_X(f) = 0, \mathsf{Lip}_d(f) \leqslant 1\}$ is compact — since X is finite, this set is actually closed and bounded in the finite dimensional space $C(X)$. Let $B > 0$ so that if $\mathsf{Lip}_d(f) \leqslant 1$ and $\tau_X(f) = 0$ then $\|f\|_{C(X)} \leqslant B$.

Now if $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}(a) \leqslant 1$ and $\tau(a) = 0$ then $\mathsf{Lip}_d \circ \rho^{-1}(\mathbb{E}(a)) \leqslant 1$ and $\tau_X(\rho^{-1}(\mathbb{E}(a))) = \tau \circ \mathbb{E}(a) = \tau(a) = 0$. Thus $\|\mathbb{E}(a)\|_{\mathfrak{A}} \leqslant B$. Now, $\|a\|_{\mathfrak{A}} \leqslant \|a - \mathbb{E}(a)\|_{\mathfrak{A}} + \|\mathbb{E}(a)\|_{\mathfrak{A}} \leqslant \beta + B$. So:

$$\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leqslant 1, \tau(a) = 0\} \subseteq \left\{a \in \mathfrak{sa}(\mathfrak{A}) : \|a\|_{\mathfrak{A}} \leqslant \beta + B\right\},$$

and the right-hand side is compact since $\mathfrak{A}$ is finite dimensional, so $(\mathfrak{A}, \mathsf{L})$ is a compact quantum metric space by Theorem (1.1.19).

Last, we check the quasi-Leibniz property of L. Let $a, b \in \mathrm{dom}(\mathsf{L})$ and $x, y \in X$. Since ρ is a *-isomorphism, we now compute:

$$\begin{aligned}
&\left|\rho^{-1}(\mathbb{E}(ab))(x) - \rho^{-1}(\mathbb{E}(ab))(y)\right| \\
&\leqslant \left|\rho^{-1}(\mathbb{E}(ab))(x) - \rho^{-1}(\mathbb{E}(a\mathbb{E}(b)))(x)\right| \\
&\quad + \left|\rho^{-1}(\mathbb{E}(a\mathbb{E}(b)))(x) - \rho^{-1}(\mathbb{E}(\mathbb{E}(a)b))(y)\right| \\
&\quad + \left|\rho^{-1}(\mathbb{E}(\mathbb{E}(a)b))(y) - \rho^{-1}(\mathbb{E}(ab))(y)\right| \\
&\leqslant \|\mathbb{E}(a(b - \mathbb{E}(b)))\|_{\mathfrak{A}} \\
&\quad + \left|\rho^{-1}(\mathbb{E}(a))(x)\rho^{-1}(\mathbb{E}(b))(x) - \rho^{-1}(\mathbb{E}(a))(y)\rho^{-1}\left(\mathbb{E}(b)(y)\right)\right| \\
&\quad + \|\mathbb{E}((a - \mathbb{E}(a))b)\|_{\mathfrak{A}} \\
&\leqslant \|a\|_{\mathfrak{A}}\beta\mathsf{L}(b) + \|b\|_{\mathfrak{A}}\beta\mathsf{L}(b) \\
&\quad + \left|\rho^{-1}(\mathbb{E}(a))(x)\rho^{-1}(\mathbb{E}(b))(x) - \rho^{-1}(\mathbb{E}(a))(y)\rho^{-1}(\mathbb{E}(b))(y)\right|.
\end{aligned}$$

Hence:

$$\begin{aligned}
&\mathsf{Lip}_d \circ \rho^{-1}(\mathbb{E}(ab)) \\
&= \sup\left\{\frac{\left|\rho^{-1}(\mathbb{E}(ab))(x) - \rho^{-1}(\mathbb{E}(ab))(y)\right|}{d(x,y)} : x, y \in X, x \neq y\right\} \\
&\leqslant \|a\|_{\mathfrak{A}}\frac{\beta}{\delta}\mathsf{L}(b) + \|b\|_{\mathfrak{A}}\frac{\beta}{\delta}\mathsf{L}(b) \\
&\quad + \sup\left\{\frac{\left|\rho^{-1}(\mathbb{E}(a))(x)\rho^{-1}(\mathbb{E}(b))(x) - \rho^{-1}(\mathbb{E}(a))(y)\rho^{-1}(\mathbb{E}(b))(y)\right|}{d(x,y)}\right. \\
&\qquad\qquad \left. : x, y \in X, x \neq y\right\} \qquad (2.8.4) \\
&\leqslant \frac{\beta}{\delta}(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{A}}) + \mathsf{Lip}_d(\mathbb{E}(a)\mathbb{E}(b)) \\
&\leqslant \frac{\beta}{\delta}(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{A}}) + \mathsf{Lip}_d \circ \mathbb{E}(a)\|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}}\mathsf{Lip}_d \circ \mathbb{E}(b) \\
&\leqslant \left(1 + \frac{\beta}{\delta}\right)(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{A}}).
\end{aligned}$$

From this and from [2, Lemma 3.2], it follows easily that $(\mathfrak{A}, \mathsf{L})$ is indeed a $(D, 0)$–quasi-Leibniz quantum compact metric space with $D = \max\left\{2, \left(1 + \frac{\beta}{\delta}\right)\right\}$.

We now compute an upper bound for $\Lambda((\mathfrak{A},\mathsf{L}),(C(X),\mathsf{Lip}_d))$ by exhibiting a particular bridge from $\mathfrak{A}$ to $C(X)$.

Let $\gamma = (\mathfrak{A},\mathrm{id},\rho,1_{\mathfrak{A}})$ where id is the identity *-morphism of $\mathfrak{A}$. By Definition (2.2.4), the quadruple γ is a bridge of height 0, so its length equals to its reach.

If $f \in C(X)$ and $\mathsf{Lip}_d(f) \leqslant 1$, then:

$$\frac{\left\|\rho(f) - \mathbb{E}(\rho(f))\right\|_{\mathfrak{A}}}{\beta} = 0$$

and $\mathsf{Lip}_d(\rho^{-1}(\mathbb{E}(\rho(f)))) = \mathsf{Lip}_d(f) \leqslant 1$. So $\mathsf{L}(\rho(f)) \leqslant 1$.

Now, it is immediate that $\mathsf{bn}_\gamma\left(\rho(f),f\right) = \|\rho(f)-\rho(f)\|_{\mathfrak{A}} = 0$. So:

$$\sup_{\substack{f\in C(X)\\ \mathsf{Lip}_d(f)\leqslant 1}} \inf_{\substack{a\in\mathfrak{sa}(\mathfrak{A})\\ \mathsf{L}(a)\leqslant 1}} \mathsf{bn}_\gamma(a,b) = 0.$$

If $a \in \mathfrak{A}$ with $\mathsf{L}(a) \leqslant 1$, then set $f = \rho^{-1}(\mathbb{E}(a))$. First, by definition of L, we have $\mathsf{Lip}_d(f) = \mathsf{Lip}_d(\rho^{-1}(\mathbb{E}(a))) \leqslant \mathsf{L}(a) \leqslant 1$. Second:

$$\left\|a - \rho(f)\right\|_{\mathfrak{A}} = \|a - \mathbb{E}(a)\|_{\mathfrak{A}} \leqslant \beta.$$

Thus

$$\sup_{\substack{a\in\mathfrak{sa}(\mathfrak{A})\\ \mathsf{L}(a)\leqslant 1}} \inf_{\substack{f\in C(X)\\ \mathsf{Lip}_d(f)\leqslant 1}} \mathsf{bn}_\gamma(a,b) \leqslant \beta.$$

Therefore, the reach, and thus the length, of γ is no more than β. Hence by Theorem (2.2.8), we conclude $\Lambda((\mathfrak{A},\mathsf{L}),(C(X),\mathsf{Lip}_d)) \leqslant \beta$ as desired. □

We now deduce from Theorem (2.8.11) that compact metric spaces are always limits of full matrix algebras for the quantum propinquity. A notable component of the following result is how the constant β of Theorem (2.8.11) are related to the actual geometry of the limit classical space.

Corollary 2.8.12. *If (X,d) is a compact metric space, if $Y \subseteq X$ is a finite subset of X, and if $\beta_Y \in (0,\infty)$ such that:*

$$\frac{\beta_Y}{\min\{d(x,y) : x,y\in Y, x\neq y\}} \leqslant 1$$

then there exists a $(2,0)$–quasi-Leibniz quantum compact metric space $(\mathfrak{A},\mathsf{L})$ where:

1. *$\mathfrak{A}$ is the C*-algebra of $\#Y \times \#Y$-matrices over $\mathbb{C}$ and τ is the unique tracial state on $\mathfrak{A}$,*

2. *with $C(Y)$ identified with the diagonal C*-subalgebra of $\mathfrak{A}$ given by a unital *-isomorphism ρ with domain $C(Y)$ and $\mathbb{E}_Y$, the unique τ-preserving conditional expectation of $\mathfrak{A}$ onto $\rho(C(Y))$, the L-seminorm L is given for all $a \in \mathfrak{A}$ by:*

$$\mathsf{L}(a) = \max\left\{\frac{\|a-\mathbb{E}_Y(a)\|_{\mathfrak{A}}}{\beta_Y}, \mathsf{Lip}_d\circ\rho^{-1}(\mathbb{E}_Y(a))\right\}, \qquad (2.8.5)$$

and

3. $\Lambda((\mathfrak{A}, \mathsf{L}), (C(X), \mathsf{Lip}_d)) \leqslant \mathsf{Haus}[d](X, Y) + \beta_Y$.

Proof. Set $\delta = \min\{d(x, y) : x, y \in Y, x \neq y\}$. By Theorem (2.8.11), the compact quantum metric space $(\mathfrak{A}, \mathsf{L})$ is$(2, 0)$-quasi-Leibniz since $1 + \frac{\beta_Y}{\delta} \leqslant 2$ and:

$$\Lambda((\mathfrak{A}, \mathsf{L}), (C(Y), \mathsf{Lip}_d)) \leqslant \beta_Y.$$

Thus:

$$\begin{aligned}\Lambda((\mathfrak{A}, \mathsf{L}), (C(X), \mathsf{Lip}_d)) \leqslant \\ \Lambda((\mathfrak{A}, \mathsf{L}), (C(Y), \mathsf{Lip}_d)) + \Lambda((C(Y), \mathsf{Lip}_d), ((C(X), \mathsf{Lip}_d))) \\ \leqslant \beta_Y + \mathsf{Haus}[d](X, Y).\end{aligned}$$

This concludes our proof. □

Corollary 2.8.13. *Any compact metric space (X, d) is the limit for the $(2, 0)$-propinquity of sequences of $(2, 0)$–quasi-Leibniz quantum compact metric spaces consisting of full matrix algebras.*

Proof. We simply apply Corollary (3.4.5) to any sequence $(X_n)_{n\in\mathbb{N}}$ of finite subsets of X with $\lim_{n\to\infty} \mathsf{Haus}[d](X, X_n) = 0$, which always exists since (X, d) is compact, and to $\left(\beta_{X_n}\right)_{n\in\mathbb{N}} = \left(\frac{1}{n}\min\{d(x, y) : x, y \in X_n, x \neq y\}\right)_{n\in\mathbb{N}}$. □

As a corollary, we obtain the following:

Theorem 2.8.14. *The class of classical compact metric spaces is closed and nowhere dense for the $(2, 0)$-propinquity.*

Proof. We saw in Theorem (2.6.2) that this class is closed for the propinquity. We note in passing this could also be checked "directly" using target sets rather than by a compactness argument.

Now, fix a compact metric space (X, d). by Corollary (2.8.13), for all $\varepsilon > 0$, there exists a $(2, 0)$-quantum compact metric space $(\mathfrak{A}, \mathsf{L})$ such that $\Lambda_{2,0}(F(X, d), (\mathfrak{A}, \mathsf{L})) < \varepsilon$. So every open ball of center (X, d) contains a noncommutative quantum compact metric space, so the class of classical compact metric spaces is nowhere dense. □

Chapter Three

Spectral Triples

3.1 METRIC SPECTRAL TRIPLES

Definition 3.1.1 ([11]). A *spectral triple* $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a triple consisting of a unital C*-algebra $\mathfrak{A}$, given with an implicit unital *-representation on a Hilbert space $\mathscr{H}$, together with a self-adjoint operator $\not{D}$, defined on a dense subspace $\operatorname{dom}(\not{D})$ of $\mathscr{H}$, such that

- $D + i$ has a compact inverse,
- the space
$$\mathfrak{A}_{\not{D}} := \{a \in \mathfrak{sa}(\mathfrak{A}) : a \cdot \operatorname{dom}(\not{D}) \subseteq \operatorname{dom}(\not{D}),\ [\not{D}, a] \text{ is bounded}\}$$
is dense in $\mathfrak{A}$.

The operator $\not{D}$ is called the *Dirac operator* of the spectral triple $(\mathfrak{A}, \mathscr{H}, \not{D})$.

A spectral triple $(\mathfrak{A}, \mathscr{H}, \not{D})$ induces, in particular, a (possibly ∞-valued) pseudo-metric on the state space of the underlying C*-algebra $\mathfrak{A}$, called the *Connes' distance*, defined, between any two states φ, ψ of $\mathfrak{A}$, by

$$\mathsf{mk}_{\not{D}}(\varphi, \psi) = \sup\left\{|\varphi(a) - \psi(a)| : a \in \mathfrak{A}_{\not{D}}, {|||[\not{D}, a]|||}_{\mathscr{H}} \leqslant 1\right\}.$$

Connes proved in [11] that, when $\mathfrak{A} = C(M)$ is the C*-algebra of $\mathbb{C}$-valued continuous over a connected, compact spin Riemannian manifold M, and when $\not{D}$ is the Dirac operator acting on the square integrable sections of the spinor bundle over M, the Connes' metric $\mathsf{mk}_{\not{D}}$ restricts to the usual path metric over M induced by its Riemannian metric. More generally, Connes' distance suggests a way to study the metric properties of noncommutative spaces, as it makes sense whether $\mathfrak{A}$ is commutative or not.

Let us now ask a natural question. If $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a spectral triple, then we can set $\mathsf{L}(a) := {|||[\not{D}, a]|||}_{\mathscr{H}}$ for all $a \in \mathfrak{sa}(\mathfrak{A})$ for which this expression makes sense and is finite. Then Connes' pseudo-distance is simply mk_{L}. When is $(\mathfrak{A}, \mathsf{L})$ a quantum compact metric space ?

Notation 3.1.2. If $T : E \to F$ is a continuous linear operator from a normed vector space E to a normed vector space F, then the norm of T is denoted by $|||T|||_F^E$; if $E = F$, then we simply write $|||T|||_E$.

Definition 3.1.3. A spectral triple $(\mathfrak{A}, \mathscr{H}, D\!\!\!/)$ is *metric* when, setting:

$$\mathrm{dom}(\mathsf{L}) = \{a \in \mathfrak{sa}(\mathfrak{A}) : a \cdot \mathrm{dom}(D\!\!\!/) \subseteq \mathrm{dom}(D\!\!\!/),\ [D\!\!\!/, a] \text{ is bounded }\}$$

and

$$\forall a \in \mathrm{dom}(\mathsf{L}) \quad \mathsf{L}(a) = |||[D\!\!\!/, a]|||_{\mathscr{H}},$$

then $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space.

Examples of metric spectral triples include, once again, quantum tori [52, 37], hyperbolic [51] and nilpotent [8] group C*-algebras, certain C*-crossed-products [23], certain fractals [9, 32], Podès spheres [1], curved quantum tori [35], among others. Certain examples motivated in physics can be found in [45].

Proposition 3.1.4. *Let $(\mathfrak{A}, \mathscr{H}, D)$ be a spectral triple. The spectral triple $(\mathfrak{A}, \mathscr{H}, D)$ is metric if and only if $(\mathfrak{A}, \mathsf{L}_D)$ is a Leibniz quantum compact metric space.*

Proof. If $(\mathfrak{A}, \mathsf{L}_D)$ is a Leibniz quantum compact metric space, then by Definition (3.1.3), the spectral triple $(\mathfrak{A}, \mathscr{H}, D)$ is metric.

Let us now assume that $(\mathfrak{A}, \mathscr{H}, D)$ is a metric spectral triple. The domain of L_D is:

$$\left\{a \in \mathfrak{sa}(\mathfrak{A}) : a \cdot \mathrm{dom}(D) \subseteq \mathrm{dom}(D) \text{ and } |||[D, a]|||_{\mathscr{H}} < \infty\right\}.$$

By Definition (3.1.1), the set:

$$\mathscr{D} = \left\{a \in \mathfrak{A} : a \cdot \mathrm{dom}(D) \subseteq \mathrm{dom}(D) \text{ and } |||[D, a]|||_{\mathscr{H}} < \infty\right\}$$

is norm dense in $\mathfrak{A}$. If $a \in \mathfrak{sa}(\mathfrak{A})$, then there exists $(a_n)_{n\in\mathbb{N}}$ in $\mathscr{D}^{\mathbb{N}}$ converging to a in norm. Now, we prove that if $b \in \mathscr{D}$ then $b^* \in \mathscr{D}$ as well. Let $b \in \mathscr{D}$. if $\xi, \zeta \in \mathrm{dom}(D)$, then:

$$\begin{aligned}
\langle b^*\xi, D\zeta\rangle_{\mathscr{H}} &= \langle \xi, bD\zeta\rangle_{\mathscr{H}} \\
&= \langle \xi, Db\zeta\rangle_{\mathscr{H}} - \langle \xi, [D, b]\zeta\rangle_{\mathscr{H}} \\
&= \langle D\xi, b\zeta\rangle_{\mathscr{H}} - \langle \xi, [D, b]\zeta\rangle_{\mathscr{H}}.
\end{aligned}$$

Now, since $\xi \in \mathrm{dom}(D)$, the linear map $\zeta \in \mathrm{dom}(D) \mapsto \langle D\xi, b\zeta\rangle_{\mathscr{H}}$ is continuous, and since $[D, b]$ is bounded, the linear map $\zeta \in \mathrm{dom}(D) \mapsto \langle \xi, [D, b]\zeta\rangle_{\mathscr{H}}$ is also continuous. Hence $\zeta \in \mathscr{H} \mapsto \langle b^*\xi, D\zeta\rangle_{\mathscr{H}}$ is continuous, and thus $b^*\xi \in \mathrm{dom}(D^*) = \mathrm{dom}(D)$. Now, on $\mathrm{dom}(D)$, we observe that $[D, b^*] = Db^* - b^*D = (bD - Db)^* = (-[D, b])^*$ as D is self-adjoint, so $b^* \in \mathscr{D}$.

It is immediate to check that $\mathscr{D}$ is a linear space, and thus in particular, for all $n \in \mathbb{N}$, we have $\Re a_n = \frac{a_n + a_n^*}{2} \in \mathrm{dom}(\mathsf{L}_D)$, and of course as $a \in \mathfrak{sa}(\mathfrak{A})$, we have by continuity of $\Re$ that $a = \Re a = \lim_{n\to\infty} \Re a_n$, thus proving that $\mathrm{dom}(\mathsf{L}_D)$ is dense in $\mathfrak{sa}(\mathfrak{A})$.

By Definition (3.1.3), the Monge-Kantorovich metric $\mathsf{mk}_{\mathsf{L}_D}$ metrizes the weak* topology. In particular, as a metric , it is finite between any two states of $\mathfrak{A}$. Let $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_D(a) = 0$. Let $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$. We have, by Definition (1.1.1):

$$0 \leqslant \left|\varphi(a) - \psi(a)\right| \leqslant \mathsf{L}_D(a)\mathsf{mk}_{\mathsf{L}_D}(\varphi, \psi) = 0$$

and thus $\varphi(a-\psi(a)1_{\mathfrak{A}}) = 0$ for all $\varphi,\psi \in \mathscr{S}(\mathfrak{A})$. Thus (as $a \in \mathfrak{sa}(\mathfrak{A})$), if we fix $\psi \in \mathscr{S}(\mathfrak{A})$:

$$\left\| a-\psi(a)1_{\mathfrak{A}}\right\|_{\mathfrak{A}} = \sup_{\varphi\in\mathscr{S}(\mathfrak{A})} |\varphi(a-\psi(a)1_{\mathfrak{A}})| = 0$$

so $a = \psi(a)1_{\mathfrak{A}}$, i.e. $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_D(a) = 0\} \subseteq \mathbb{R}1_{\mathfrak{A}}$. On the other hand, $\mathsf{L}_D(1_{\mathfrak{A}}) = 0$ by construction, so $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_D(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$, as desired.

We now check that L_D is lower semicontinuous. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{dom}(\mathsf{L}_D)$ with $\mathsf{L}_D(a_n) \leqslant 1$ converging in norm to $a \in \mathfrak{sa}(\mathfrak{A})$. Let $\xi \in \mathrm{dom}(D)$ and let $\zeta \in \mathrm{dom}(D)$. For any $n \in \mathbb{N}$:

$$\begin{aligned}\langle a_n\xi, D\zeta\rangle_{\mathscr{H}} &= \langle \xi, a_n D\zeta\rangle_{\mathscr{H}}\\ &= \langle \xi, Da_n\zeta\rangle_{\mathscr{H}} - \langle \xi, [D,a_n]\zeta\rangle_{\mathscr{H}}\\ &= \langle D\xi, a_n\zeta\rangle_{\mathscr{H}} - \langle \xi, [D,a_n]\zeta\rangle_{\mathscr{H}}\end{aligned}$$

and therefore:

$$\begin{aligned}\left|\langle a\xi, D\zeta\rangle_{\mathscr{H}}\right| &= \lim_{n\to\infty}\left|\langle a_n\xi, D\zeta\rangle_{\mathscr{H}}\right|\\ &\leqslant \limsup_{n\to\infty}\left(\left|\langle D\xi, a_n\zeta\rangle_{\mathscr{H}}\right| + \left|\langle \xi, [D,a_n]\zeta\rangle_{\mathscr{H}}\right|\right)\\ &\leqslant \left|\langle D\xi, a\zeta\rangle_{\mathscr{H}}\right| + \|\xi\|_{\mathscr{H}}\|\zeta\|_{\mathscr{H}}\\ &\leqslant \|\zeta\|_{\mathscr{H}}\left(\|D\xi\|_{\mathscr{H}}\|a\|_{\mathfrak{A}} + \|\xi\|_{\mathscr{H}}\right).\end{aligned}$$

So the function $\zeta \in \mathrm{dom}(D) \mapsto \langle a\xi, D\zeta\rangle_{\mathscr{H}}$ is continuous, and thus $a\xi \in \mathrm{dom}(D)$. Thus $a\cdot\mathrm{dom}(D) \subseteq \mathrm{dom}(D)$ as $\xi \in \mathrm{dom}(D)$ was arbitrary. We can therefore apply [53, Proposition 3.7], whose argument we now briefly recall. If $\xi,\zeta \in \mathrm{dom}(D)$ with $\|\xi\|_{\mathscr{H}} \leqslant 1$ and $\|\zeta\|_{\mathscr{H}} \leqslant 1$, then:

$$\begin{aligned}1 \geqslant \left|\langle [D,a_n]\xi,\zeta\rangle_{\mathscr{H}}\right| &= \left|\langle a_n\xi, D\zeta\rangle_{\mathscr{H}} - \langle D\xi, a_n\zeta\rangle_{\mathscr{H}}\right|\\ &\xrightarrow{n\to\infty} \left|\langle a\xi, D\zeta\rangle_{\mathscr{H}} - \langle D\xi, a\zeta\rangle_{\mathscr{H}}\right| = \left|\langle [D,a]\xi,\zeta\rangle_{\mathscr{H}}\right|. \quad (3.1.1)\end{aligned}$$

Since $\mathrm{dom}(D)$ is dense in $\mathscr{H}$ and since, by Expression (3.1.1), for all $\xi,\zeta \in \mathrm{dom}(D)$, we have proven that $\left|\langle [D,a]\xi,\zeta\rangle_{\mathscr{H}}\right| \leqslant \|\xi\|_{\mathscr{H}}\|\zeta\|_{\mathscr{H}}$, we conclude that $[D,a]$ is bounded with norm 1 on $\mathrm{dom}(D)$, and thus extends to a bounded operator of norm at most 1 on $\mathscr{H}$.

Thus $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_D(a) \leqslant 1\}$ is indeed normed closed. As L_D is a seminorm, this implies that it is lower semi-continuous with respect to $\|\cdot\|_{\mathfrak{A}}$.

Last, L_D satisfies the Leibniz inequality since it is the norm of a derivation. First, we note that $\mathscr{D}$ is indeed an algebra. If $a,b \in \mathscr{D}$ then, first, since $b\cdot\mathrm{dom}(D) \subseteq \mathrm{dom}(D)$, we also have $ab\cdot\mathrm{dom}(D) \subseteq a\cdot\mathrm{dom}(D) \subseteq \mathrm{dom}(D)$. Moreover, if $\xi,\zeta \in \mathrm{dom}(D)$, then:

$$\begin{aligned}\langle Dab\xi - abD\xi,\zeta\rangle_{\mathscr{H}} &= \langle Dab\xi - aDb\xi,\zeta\rangle_{\mathscr{H}} + \langle aDb\xi - abD\xi,\zeta\rangle_{\mathscr{H}}\\ &= \langle [D,a]b\xi,\zeta\rangle_{\mathscr{H}} + \langle a[D,b]\xi,\zeta\rangle_{\mathscr{H}}\end{aligned}$$

and thus, as operators on $\operatorname{dom}(D)$, we conclude $[D,ab] = a[D,b]+[D,a]b$. Therefore, for all $a,b \in \operatorname{dom}(\mathsf{L}_D)$:

$$\begin{aligned} \left|\!\left|\!\left|[D,ab]\right|\!\right|\!\right|_{\mathscr{H}} &= \left|\!\left|\!\left|[D,a]b + a[D,b]\right|\!\right|\!\right|_{\mathscr{H}} \\ &\leqslant \left|\!\left|\!\left|[D,a]\right|\!\right|\!\right|_{\mathscr{H}} \|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}} \left|\!\left|\!\left|[D,b]\right|\!\right|\!\right|_{\mathscr{H}} \\ &= \mathsf{L}_D(a)\|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}} \mathsf{L}_D(b). \end{aligned}$$

Therefore, we conclude, for all $a,b \in \operatorname{dom}(\mathsf{L}_D)$:

$$\begin{aligned} \mathsf{L}\left(\frac{ab+ba}{2}\right) &= \left|\!\left|\!\left|\left[D, \frac{ab+ba}{2}\right]\right|\!\right|\!\right|_{\mathscr{H}} \\ &\leqslant \frac{1}{2}\left(\left|\!\left|\!\left|[D,ab]\right|\!\right|\!\right|_{\mathscr{H}} + \left|\!\left|\!\left|[D,ba]\right|\!\right|\!\right|_{\mathscr{H}}\right) \\ &\leqslant \mathsf{L}_D(a)\|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}} \mathsf{L}_D(b). \end{aligned}$$

A similar argument shows that $\mathsf{L}_D\left(\frac{ab-ba}{2i}\right) \leqslant \mathsf{L}_D(a)\|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}} \mathsf{L}_D(b)$. It follows that $(\mathfrak{A}, \mathsf{L}_D)$ is a quantum compact metric space. □

A strong notion of equivalence between spectral triples is given by:

Definition 3.1.5. Two spectral triples $(\mathfrak{A}, \mathscr{H}_{\mathfrak{A}}, D_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathscr{H}_{\mathfrak{B}}, D_{\mathfrak{B}})$ are *equivalent* when there exists a unitary U from $\mathscr{H}_{\mathfrak{A}}$ to $\mathscr{H}_{\mathfrak{B}}$ and a *-isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$, such that

$$U\operatorname{dom}(D_{\mathfrak{A}}) = \operatorname{dom}(D_{\mathfrak{B}}) \text{ and } U^* D_{\mathfrak{B}} U = D_{\mathfrak{A}} \text{ over } \operatorname{dom}(D_{\mathfrak{A}}),$$

and

$$\forall \omega \in \mathscr{H}_{\mathfrak{B}}, a \in \mathfrak{A} \quad \theta(a)\omega = (UaU^*)\omega.$$

Proposition 3.1.6. *If $(\mathfrak{A}, \mathscr{H}_{\mathfrak{A}}, D_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathscr{H}_{\mathfrak{B}}, D_{\mathfrak{B}})$ are two equivalent metric spectral triples, then $(\mathfrak{A}, \mathsf{L}_{D_{\mathfrak{A}}})$ and $(\mathfrak{B}, \mathsf{L}_{D_{\mathfrak{B}}})$ are fully quantum isometric.*

Notation 3.1.7. If T is an invertible operator on a Hilbert space $\mathscr{H}$, then $\operatorname{Ad}_T(A) = TAT^{-1}$ for all operators A (bounded or not, up to adjusting the domain).

Proof. Let $U : \mathscr{H}_{\mathfrak{A}} \to \mathscr{H}_{\mathfrak{B}}$ be unitary and $\theta : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be a *-isomorphism such that $\operatorname{Ad}_U D_{\mathfrak{A}} = D_{\mathfrak{B}}$ (including the fact that $U\operatorname{dom}(D_{\mathfrak{A}}) = \operatorname{dom}(D_{\mathfrak{B}})$), and $UaU^* = \theta(a)$ for all $a \in \mathfrak{A}$. If $a \in \operatorname{dom}\left(\mathsf{L}_{D_{\mathfrak{A}}}\right)$ then $a \cdot \operatorname{dom}(D_{\mathfrak{A}}) \subseteq \operatorname{dom}(D_{\mathfrak{A}})$, and $[D_{\mathfrak{A}}, a]$ is bounded. Now, if $\xi \in \operatorname{dom}(D_{\mathfrak{B}})$, then $U^*\xi \in \operatorname{dom}(D_{\mathfrak{A}})$, and therefore,

$UaU^*\xi \in \mathrm{dom}(D_{\mathfrak{B}})$. Moreover:

$$\begin{aligned}
\mathsf{L}_{D_{\mathfrak{A}}}(a) &= |||[D_{\mathfrak{A}}, a]|||_{\mathscr{H}_{\mathfrak{A}}} \\
&= |||[D_{\mathfrak{A}}, a]|||_{\mathrm{dom}(D_{\mathfrak{A}})} \\
&= \left|\left|\left|[U^* D_{\mathfrak{B}} U, a]\right|\right|\right|_{\mathrm{dom}(D_A)} \\
&= \left|\left|\left|U^* D_{\mathfrak{B}} U a - a U^* D_{\mathfrak{B}} U\right|\right|\right|_{\mathrm{dom}(D_{\mathfrak{A}})} \\
&= \left|\left|\left|U^* \left(D_{\mathfrak{B}} UaU^* - UaU^* D_{\mathfrak{A}}\right) U\right|\right|\right|_{\mathrm{dom}(D_{\mathfrak{A}})} \\
&= |||D_{\mathfrak{B}}\theta(a) - \theta(a) D_{\mathfrak{B}}|||_{\mathrm{dom}(D_{\mathfrak{B}})} \\
&= |||[D_{\mathfrak{B}}, \theta(a)]|||_{\mathscr{H}_{\mathfrak{B}}} \\
&= \mathsf{L}_{D_{\mathfrak{B}}} \circ \theta(a).
\end{aligned}$$

Thus $\theta(a) \in \mathrm{dom}\left(\mathsf{L}_{D_{\mathfrak{B}}}\right)$ and $\mathsf{L}_{D_{\mathfrak{B}}} \circ \theta(a) = \mathsf{L}_{D_{\mathfrak{A}}}(a)$. In particular, $\theta(\mathrm{dom}\left(\mathsf{L}_{D_{\mathfrak{A}}}\right)) \subseteq \mathrm{dom}\left(\mathsf{L}_{D_{\mathfrak{B}}}\right)$.

By symmetry, if $b \in \mathrm{dom}\left(\mathsf{L}_{D_{\mathfrak{B}}}\right)$, then $\theta^{-1}(b) \in \mathsf{L}_{D_{\mathfrak{A}}}$ with $\mathsf{L}_{D_{\mathfrak{A}}} \circ \theta^{-1}(b) = \mathsf{L}_{D_{\mathfrak{B}}}(b)$.

If $a \notin \mathrm{dom}\left(\mathsf{L}_{D_{\mathfrak{A}}}\right)$, yet $\theta(a) \in \mathrm{dom}\left(\mathsf{L}_{D_{\mathfrak{B}}}\right)$, then we would have, by the observation above, that $a = \theta^{-1}(\theta(a)) \in \mathrm{dom}\left(\mathsf{L}_{D_{\mathfrak{A}}}\right)$, an obvious contradiction. So $\theta(\mathfrak{sa}(\mathfrak{A}) \setminus \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathfrak{sa}(\mathfrak{B}) \setminus \mathrm{dom}\left(\mathsf{L}_{D_{\mathfrak{B}}}\right)$. Therefore, $\theta(\mathrm{dom}\left(\mathsf{L}_{D_{\mathfrak{A}}}\right)) = \mathrm{dom}\left(\mathsf{L}_{D_{\mathfrak{B}}}\right)$.

Thus θ is a full quantum isometry from $(\mathfrak{A}, \mathsf{L}_{D_{\mathfrak{A}}})$ to $(\mathfrak{B}, \mathsf{L}_{D_{\mathfrak{B}}})$. □

We now want to find a metric description of a spectral triple. If $(\mathfrak{A}, \mathscr{H}, \mathbb{D})$ is a metric spectral triple, and if we set:

- $\mathrm{dom}(\mathsf{L}_{\mathbb{D}}) := \{a \in \mathfrak{sa}(\mathfrak{A}) : a\,\mathrm{dom}(\mathbb{D}) \subseteq \mathrm{dom}(\mathbb{D}), [\mathbb{D}, a] \text{ is bounded}\}$,
- $\forall a \in \mathrm{dom}(\mathsf{L}_{\mathbb{D}}) \quad \mathsf{L}(a) := |||[\mathbb{D}, a]|||_{\mathscr{H}}$,
- $\forall \xi \in \mathrm{dom}(\mathbb{D}) \quad \mathsf{DN}(\xi) := \|\xi\|_{\mathscr{H}} + \|\mathbb{D}\xi\|_{\mathscr{H}}$,

then $(\mathscr{H}, \mathsf{DN}, \mathbb{C}, 0, \mathfrak{A}, \mathsf{L}_{\mathbb{D}})$ is an example of the following general structure.

Definition 3.1.8. A *(F,K,G,H)-metrical C*-correspondence* $(\mathscr{M}, \mathsf{DN}, \mathfrak{A}, \mathsf{L}, \mathfrak{B}, \mathsf{S})$, where $F, G \geqslant 1$, $H \geqslant 2$, and $K > 0$, is given by two (F,K)-quantum compact metric spaces $(\mathfrak{A}, \mathsf{L})$ and $(\mathfrak{B}, \mathsf{S})$, an $\mathfrak{A}$-$\mathfrak{B}$ C*-correspondence $(\mathscr{M}, \mathfrak{A}, \mathfrak{B})$, and a norm DN defined on a dense $\mathbb{C}$-subspace $\mathrm{dom}(\mathsf{TN})$ of $\mathscr{M}$, such that

1. $\forall \omega \in \mathrm{dom}(\mathsf{DN}) \quad \mathsf{DN}(\omega) \geqslant \|\omega\|_{\mathscr{M}} := \sqrt{\left\|\langle \omega, \omega \rangle_{\mathscr{M}}\right\|_{\mathfrak{B}}}$,
2. $\{\omega \in \mathrm{dom}(\mathsf{DN}) : \mathsf{DN}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
3. for all $a \in \mathrm{dom}(\mathsf{L})$ and $\omega \in \mathrm{dom}(\mathsf{TN})$,
$$\mathsf{DN}(a\omega) \leqslant G(\|a\|_{\mathfrak{A}} + \mathsf{L}(a))\mathsf{DN}(\omega),$$
4. for all $\omega, \eta \in \mathrm{dom}(\mathsf{DN})$,
$$\mathsf{S}(\langle \omega, \eta \rangle_{\mathscr{M}}) \leqslant H\,\mathsf{DN}(\omega)\mathsf{DN}(\eta).$$

Convention 3.1.9. In this work, we *fix* $F \geqslant 1$, $K \geqslant 0$, $H \geqslant 2$ and $G \geqslant 1$ all throughout the paper. *All* quantum compact metric spaces will be assumed to be in the class of (F, K)-quantum compact metric spaces and all metrical C*-correspondences will be assume to be in the class of (F, K, G, H)-metrical C*-correspondences, unless otherwise specified.

Theorem 3.1.10 ([47])**.** *If $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a metric spectral triple, if we set:*

$$\forall \xi \in \mathrm{dom}(\not{D}) \quad \mathsf{DN}_{\not{D}}(\xi) = \|\xi\|_{\mathscr{H}} + \|\not{D}\xi\|_{\mathscr{H}},$$

and if $\mathrm{dom}(\mathsf{L}_{\not{D}}) = \{a \in \mathfrak{sa}(\mathfrak{A}) : a\mathrm{dom}(\not{D}) \subseteq \mathrm{dom}(\not{D}),\ [\not{D}, a]$ *is bounded*$\}$ *and*

$$\forall a \in \mathrm{dom}(\mathsf{L}_{\not{D}}) \quad \mathsf{L}_{\not{D}}(a) = |||[\not{D}, a]|||_{\mathscr{H}},$$

then

$$\mathrm{metCor}(\mathfrak{A}, \mathscr{H}, \not{D}) := (\mathscr{H}, \mathsf{DN}_{\not{D}}, \mathfrak{A}, \mathsf{L}_{\not{D}}, \mathbb{C}, 0)$$

is a metrical quantum vector bundle(with $F = 1, K = 0, G = 1, H = 2$).

Theorem 3.1.11. *Let $(\mathfrak{A}, \mathscr{H}, D)$ be a metric spectral triple. If for all $a \in \mathfrak{A}$ such that $a\,\mathrm{dom}(D) \subseteq D$ and $[D, a]$ is bounded on $\mathrm{dom}(D)$, we set:*

$$\mathsf{L}_D(a) = |||[D, \pi(a)]|||_{\mathscr{H}},$$

and, for all $\xi \in \mathrm{dom}(D)$, we set:

$$\mathsf{DN}(\xi) = \|\xi\|_{\mathscr{H}} + \|D\xi\|_{\mathscr{H}},$$

then $(\mathscr{H}, \mathsf{DN}, \mathfrak{A}, \mathsf{L}_D, \mathbb{C}, 0)$ is a Leibniz metrical quantum vector bundle, which we denote by $\mathrm{mvb}(\mathfrak{A}, \mathscr{H}, D)$.

Proof. For any $a \in \mathrm{dom}(\mathsf{L}_D)$ and $\xi \in \mathrm{dom}(D)$, we compute:

$$\begin{aligned}
\langle Da\xi, Da\xi\rangle_{\mathscr{H}} &= \langle Da\xi - aD\xi, Da\xi\rangle_{\mathscr{H}} + \langle aD\xi, Da\xi\rangle_{\mathscr{H}} \\
&= \langle [D,a]\xi, Da\xi\rangle_{\mathscr{H}} + \langle aD\xi, Da\xi\rangle_{\mathscr{H}} \\
&= \langle [D,a]\xi, [D,a]\xi\rangle_{\mathscr{H}} + \langle [D,a]\xi, aD\xi\rangle_{\mathscr{H}} \\
&\quad + \langle aD\xi, Da\xi\rangle_{\mathscr{H}} \\
&= \langle [D,a]\xi, [D,a]\xi\rangle_{\mathscr{H}} + \langle [D,a]\xi, aD\xi\rangle_{\mathscr{H}} \\
&\quad + \langle aD\xi, [D,a]\xi\rangle_{\mathscr{H}} + \langle aD\xi, aD\xi\rangle_{\mathscr{H}} \\
&= \langle [D,a]\xi, [D,a]\xi\rangle_{\mathscr{H}} + 2\Re\langle [D,a]\xi, aD\xi\rangle_{\mathscr{H}} \\
&\quad + \langle aD\xi, aD\xi\rangle_{\mathscr{H}} \\
&\leqslant \|[D,a]\xi\|_{\mathscr{H}}^2 + 2\,\|[D,a]\xi\|_{\mathscr{H}}\,\|a\|_{\mathfrak{A}}\,\|D\xi\|_{\mathscr{H}} + \|a\|_{\mathfrak{A}}^2\,\|D\xi\|_{\mathscr{H}}^2 \\
&= \left(\|[D,a]\xi\|_{\mathscr{H}} + \|a\|_{\mathfrak{A}}\,\|D\xi\|_{\mathscr{H}}\right)^2 \\
&\leqslant \left(\mathsf{L}_D(a)\,\|\xi\|_{\mathscr{H}} + \|a\|_{\mathfrak{A}}\,\mathsf{DN}(\xi)\right)^2.
\end{aligned}$$

Hence, $\|Da\xi\|_{\mathscr{H}} \leqslant \mathsf{L}_D(a)\,\|\xi\|_{\mathscr{H}} + \|a\|_{\mathfrak{A}}\,\|D\xi\|_{\mathscr{H}}$. Now, since $\|a\xi\|_{\mathscr{H}} \leqslant \|a\|_{\mathfrak{A}}\,\|\xi\|_{\mathscr{H}}$, we conclude that $\mathsf{DN}(a\xi) \leqslant \mathsf{L}_D(a)\,\|\xi\|_{\mathscr{H}} + \|a\|_{\mathfrak{A}}\,\mathsf{DN}(\xi) \leqslant (\mathsf{L}_D(a) + \|a\|_{\mathfrak{A}})\mathsf{DN}(\xi)$.

Now, $\mathscr{H}$ is a Hilbert $\mathbb{C}$-module, and $(\mathbb{C},0)$ is a Leibniz quantum compact metric space (the only possible one with C*-algebra $\mathbb{C} = C(\{0\}))$. Therefore, $(\mathscr{H},\mathsf{DN},\mathfrak{A},\mathsf{L},\mathbb{C},0)$ has all the properties of a Leibniz metrical quantum vector bundle, as long as we prove the compactness of the unit ball of DN.

Let $\xi \in \mathrm{dom}(D)$ with $\mathsf{DN}(\xi) \leqslant 1$. By construction, $\|(D+i)\xi\|_{\mathscr{H}} \leqslant \|D\xi\|_{\mathscr{H}} + \|\xi\|_{\mathscr{H}} \leqslant 1$. By definition, $D+i$ has a compact inverse, which we denote by K. We then have:

$$\begin{aligned}\left\{\xi \in \mathscr{H} : \mathsf{DN}(\xi) \leqslant 1\right\} &= K\left\{(D+i)\xi : \xi \in \mathscr{H}, \mathsf{DN}(\xi) \leqslant 1\right\}\\ &\subseteq K\left\{\xi \in \mathscr{H} : \|\xi\|_{\mathscr{H}} \leqslant 1\right\}\end{aligned}$$

and, as K is compact, the set $K\left\{\xi \in \mathscr{H} : \|\xi\|_{\mathscr{H}} \leqslant 1\right\}$, and therefore, the unit ball of DN, are totally bounded in $\mathscr{H}$.

It remains to show that DN is lower semicontinuous. We thus now prove that the unit ball of DN is closed in $\|\cdot\|_{\mathscr{H}}$.

Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{dom}(D)$ converging to ξ in $\mathscr{H}$ and with $\mathsf{DN}(\xi_n) \leqslant 1$ for all $n \in \mathbb{N}$. Let $\eta \in \mathrm{dom}(D)$. We compute:

$$\begin{aligned}\left|\langle \xi, D\eta\rangle_{\mathscr{H}}\right| &= \lim_{n\to\infty}\left|\langle \xi_n, D\eta\rangle_{\mathscr{H}}\right|\\ &= \lim_{n\to\infty}\left|\langle D\xi_n, \eta\rangle_{\mathscr{H}}\right|\\ &\leqslant \limsup_{n\to\infty}\|D\xi_n\|_{\mathscr{H}}\left\|\eta\right\|_{\mathscr{H}}\\ &\leqslant \limsup_{n\to\infty}\left(1-\|\xi_n\|_{\mathscr{H}}\right)\left\|\eta\right\|_{\mathscr{H}}\\ &= \left(1-\|\xi\|_{\mathscr{H}}\right)\left\|\eta\right\|_{\mathscr{H}}.\end{aligned}$$

Therefore, the map $\eta \in \mathrm{dom}(D) \mapsto \langle \xi, D\eta\rangle_{\mathscr{H}}$ is continuous. Hence $\xi \in \mathrm{dom}(D^*) = \mathrm{dom}(D)$, and thus for all $\eta \in \mathrm{dom}(D)$:

$$\left|\langle D\xi, \eta\rangle_{\mathscr{H}}\right| = \left|\langle \xi, D\eta\rangle_{\mathscr{H}}\right| \leqslant \left(1-\|\xi\|_{\mathscr{H}}\right)\left\|\eta\right\|_{\mathscr{H}}.$$

Thus $\eta \in \mathrm{dom}(D) \mapsto \langle D\xi, \eta\rangle_{\mathscr{H}}$ is uniformly continuous (as a $\left(1-\|\xi\|_{\mathscr{H}}\right)$-Lipschitz function) linear map on the dense subset $\mathrm{dom}(D)$, and thus extends uniquely to $\mathscr{H}$, where it has norm $1-\|\xi\|_{\mathscr{H}}$. Therefore $\|D\xi\|_{\mathscr{H}} \leqslant 1-\|\xi\|_{\mathscr{H}}$ and thus $\mathsf{DN}(\xi) \leqslant 1$ as desired.

Thus DN is indeed a D-norm.

Hence, if $(\mathfrak{A},\mathsf{L})$ is a quantum compact metric space, we conclude that:

$$\mathrm{mvb}(\mathfrak{A},\mathscr{H},D) = (\mathscr{H},\mathsf{DN},\mathfrak{A},\mathsf{L},\mathbb{C},0)$$

is a Leibniz metrical quantum vector bundle. □

3.2 AN OVERVIEW OF THE SPECTRAL PROPINQUITY

We defined in [41, 47] an analogue of the Gromov-Hausdorff distance on the class of metrical quantum vector bundles, which, in particular, induces a first pseudo-distance on metric spectral triples, via Theorem (3.1.10). We begin the presentation of this metric with a few key concepts.

A morphism between metrical C*-correspondences is given by a triple of linear maps which satisfy a long but very natural list of algebraic and analytic properties.

Definition 3.2.1. For each $j \in \{1,2\}$, let

$$\mathbb{M}_j = \left(\mathscr{M}_j, \mathsf{DN}_j, \mathfrak{A}_j, \mathsf{L}_j, \mathfrak{B}_j, \mathsf{S}_j\right)$$

be a metrical C*-correspondence.

A *Lipschitz morphism* (Π, π, θ) from $\mathbb{M}_1$ to $\mathbb{M}_2$ is a given by:

1. a continuous $\mathbb{C}$-linear map $\Pi : \mathscr{M}_1 \to \mathscr{M}_2$,
2. a unital *-morphism $\pi : \mathfrak{A}_1 \to \mathfrak{A}_2$,
3. a unital *-morphism $\theta : \mathfrak{B}_1 \to \mathfrak{B}_2$,

such that

1. $\forall a \in \mathfrak{A} \quad \forall \omega \in \mathscr{M}_1 \quad \Pi(a\omega) = \pi(a)\Pi(\omega)$,
2. $\forall b \in \mathfrak{B} \quad \forall \omega \in \mathscr{M}_2 \quad \Pi(\omega \cdot b) = \Pi(\omega)\theta(b)$,
3. $\forall \omega, \eta \in \mathscr{M}_1 \quad \theta(\langle \omega, \eta \rangle_{\mathscr{M}_1}) = \langle \Pi(\omega), \Pi(\eta) \rangle_{\mathscr{M}_2}$,
4. $\pi(\mathrm{dom}(\mathsf{L}_1)) \subseteq \mathrm{dom}(\mathsf{L}_2)$,
5. $\theta(\mathrm{dom}(\mathsf{S}_1)) \subseteq \mathrm{dom}(\mathsf{S}_2)$,
6. $\Pi(\mathrm{dom}(\mathsf{DN}_1)) \subseteq \mathrm{dom}(\mathsf{DN}_2)$.

Definition 3.2.2. For each $j \in \{1,2\}$, let

$$\mathbb{M}_j = \left(\mathscr{M}_j, \mathsf{DN}_j, \mathfrak{A}_j, \mathsf{L}_j, \mathfrak{B}_j, \mathsf{S}_j\right)$$

be a metrical C*-correspondence.

A *modular quantum isometry* (Π, π, θ) is a Lipschitz morphism from $\mathbb{M}_1$ to $\mathbb{M}_2$ such that

1. $\forall a \in \mathrm{dom}(\mathsf{L}_2) \quad \mathsf{L}_2(a) = \inf\{\mathsf{L}_1(d) : d \in \mathrm{dom}(\mathsf{L}_1), \pi(d) = a\}$,
2. $\forall b \in \mathrm{dom}(\mathsf{S}_2) \quad \mathsf{S}_2(b) = \inf\{\mathsf{S}_1(d) : d \in \mathrm{dom}(\mathsf{S}_1), \theta(d) = b\}$,
3. $\forall \omega \in \mathrm{dom}(\mathsf{DN}_2) \quad \mathsf{DN}_2(\omega) = \inf\left\{\mathsf{DN}_1(\eta) : \eta \in \mathrm{dom}(\mathsf{DN}_1), \theta(\eta) = \omega\right\}$.

Since metrical C*-correspondences morphisms (Π, π, θ) consists, by definition, of three isometric maps Π, π and θ, they all have closed ranges in their respective codomains; when (Π, π, θ) is a quantum isometry, our definition implies that these ranges contain certain dense subspaces, and thus, Π, π and θ are in fact surjective by definition.

The definition of a distance between metrical quantum vector bundles, called the *metrical propinquity*, relies on a notion of isometric embedding called a tunnel, and defined as follows.

Definition 3.2.3. Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two metrical C*-correspondences. A *(metrical) tunnel* $\tau = (\mathbb{J}, \Pi_1, \Pi_2)$ is a triple given by a metrical C*-correspondence $\mathbb{J}$, and for each $j \in \{1,2\}$, a modular quantum isometry $\Pi_j : \mathbb{J} \mapsto \mathbb{M}_j$.

Remark 3.2.4. It is important to note that our tunnels involve (F,K,G,H)-C*-metrical correspondences only (as per Convention (3.1.9)). We will dispense calling our tunnels (F,K,G,H)-tunnels, to keep our notation simple, but it should be stressed that *fixing* (F,K,G,H) and staying within the class of (F,K,G,H)-C*-metrical correspondences is important to obtain a metric from tunnels.

To each tunnel, we can associate a number, which quantifies how far two metrical quantum vector bundles are from the perspective of this particular tunnel. This number is computed using the Hausdorff distance induced by the Monge-Kantorovich metric on the hyperspace of all closed subsets of the state space of a quantum compact metric space.

Definition 3.2.5. Let $\mathbb{M}_j = (\mathscr{M}_j, \mathsf{DN}_j, \mathfrak{A}_j, \mathsf{L}_j, \mathfrak{B}_j, \mathsf{S}_j)$ be a metrical C*-correspondence, for each $j \in \{1,2\}$. Let $\tau = (\mathbb{P}, (\Pi_1, \pi_1, \theta_1), (\Pi_2, \pi_2, \theta_2))$ be a metrical tunnel form $\mathbb{M}_1$ to $\mathbb{M}_2$, with $\mathbb{P} = (\mathscr{P}, \mathsf{TN}, \mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \mathfrak{E}, \mathsf{L}_{\mathfrak{E}})$.

The *extent* $\chi(\tau)$ of a metrical tunnel τ is

$$\chi(\tau) := \max_{j\in\{1,2\}} \max\Big\{ \mathsf{Haus}\left[\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}\right]\left(\left\{\varphi\circ\pi_j : \varphi \in \mathscr{S}(\mathfrak{A}_j)\right\}, \mathscr{S}(\mathfrak{D})\right), \\ \mathsf{Haus}\left[\mathsf{mk}_{\mathsf{S}_{\mathfrak{E}}}\right]\left(\left\{\psi\circ\theta_j : \psi \in \mathscr{S}(\mathfrak{B}_j)\right\}, \mathscr{S}(\mathfrak{E})\right)\Big\}$$

We define our distance between metrical quantum vector bundles following the model proposed by Edwards [15] and Gromov [19], as follows.

Definition 3.2.6. The *metrical propinquity* $\Lambda^{*\mathsf{met}}(\mathbb{M}_1, \mathbb{M}_2)$ between any two metrical C*-correspondences $\mathbb{M}_1$ and $\mathbb{M}_2$ is given by the real number:

$$\Lambda^{*\mathsf{met}}(\mathbb{M}_1, \mathbb{M}_2) = \inf\left\{\chi(\tau) : \tau \text{ is a tunnel from } \mathbb{M}_1 \text{ to } \mathbb{M}_2 \right\}.$$

We prove that the metrical propinquity enjoys some welcomed properties.

Theorem 3.2.7 ([41, 46])**.** *The metrical propinquity $\Lambda^{*\mathsf{met}}$ is a complete metric, up to a full quantum isometry, on the class of metrical C*-correspondences.*

The proof follows a similar strategy as the proof of Theorem (2.4.3), with an appropriate notion of target sets.

Remark 3.2.8. The metrical propinquity, as defined above, should properly be called the (F,K,G,H)-metrical propinquity, denoted by $\Lambda^{*\mathsf{met}}_{F,K,G,H}$, as it is defined on the class of (F,K,G,H)-metrical C*-correspondences (i.e. C*-correspondences with fixed Leibniz properties). However, following our Convention (3.1.9), we will omit this index and terminology, with the understanding that we decided to restrict ourselves to this class of metrical C*-correspondence from the beginning.

Remark 3.2.9. Since metrical quantum vector bundles defined via metric spectral triples, using Theorem (3.1.10), are all of the same type ($F = 1, K = 0, G = 1, H = 2$), it may appear that, for the present paper, there is no need for the larger generality allowed by introducing these numbers. However, this is not the case. While the metrical quantum vector bundles induced by metric spectral triples are all of the same type, tunnels may well involve metrical quantum vector bundles with relaxed Leibniz properties.

If $(\mathfrak{A}, \mathsf{L})$ is a quantum compact metric space, we can regard $\mathfrak{A}$ as a Hilbert module over itself, with $\langle a, b\rangle_{\mathfrak{A}} = a^* b$ for all $a, b \in \mathfrak{A}$, and we can set

$$\forall a \in \mathfrak{A} \quad \mathsf{DN}(a) = \max\{\mathsf{L}(\Re a), \mathsf{L}(\Im a)\},$$

thus associating to any quantum compact metric space the metrical C*-correspondence $(\mathfrak{A}, \mathsf{DN}, \mathbb{C}, 0, \mathfrak{A}, \mathsf{L})$. The injection thus constructed from the class of quantum compact metric spaces to the class of metrical C*-correspondences, is an homeomorphism onto its range, when the class of quantum compact metric spaces is endowed with the dual propinquity, and the class of metrical C*-correspondences is metrized with the metrical propinquity. In the present work, however, we focus on the class of metrical C*-correspondences.

If we apply the metrical propinquity to the metrical quantum vector bundles associated with metric spectral triples, then we obtain a pseudo-metric on spectral triples. This metric, informally, captures the metric information of the spectral triple. We note, in passing, that tunnels are not required to be constructed using spectral triples — in fact, this flexibility is very helpful. We thus have some preliminary notion of convergence for spectral triples.

However, distance zero between spectral triples is weaker than what we want.

In order to strengthen the metrical propinquity, we add one more ingredient. In yet another chapter of our story [42, 43, 47], we introduced covariant versions of the propinquity. Now, if $(\mathfrak{A}, \mathscr{H}, \mathbb{D})$ is a spectral triple, then it induces a natural quantum dynamics, i.e. an action of $\mathbb{R}$ on $\mathscr{H}$ by unitaries, defined for each $t \in \mathbb{R}$ by $U^t = \exp(it\mathbb{D})$.

The *spectral propinquity* is thus the covariant version of the metrical propinquity, applied to the metrical quantum vector bundle induced by a metric spectral triples *and* the associated unitary action, restricted to the monoid $[0, \infty)$. Convergence, in the sense of the propinquity, is the main matter of study of this paper. The spectral propinquity is, indeed, a distance on the class of metric spectral triples, up to unitary equivalence; we also proved some nontrivial examples of convergence for this metric [31, 45]. The covariant version of the propinquity is best explained by introducing first a useful notion of convergence of operators over metrical quantum vector bundles, itself of prime interest for this work, and new to this paper. This is the first matter which we address.

3.3 CONVERGENCE FOR FAMILY OF OPERATORS ON C*-CORRESPONDENCES

In this section, we define a distance, up to unitary equivalence, between families of operators on Hilbert spaces. Our purpose with this distance is to formalize the convergence of the bounded functional calculus for spectral triples under convergence for the propinquity, which is the core result of this paper. While, in this work, we will work with family of operators on Hilbert spaces associated with spectral triples, the definition of a metrical tunnel means that the computation of various distances will take place within the state space of metrical C*-correspondences, which is thus the framework we employ in this section.

Our first task in extending the construction of the propinquity to family of operators on modules, including to actions of monoids and groups as needed for the spectral propinquity, is to extend the Monge-Kantorovich metric of a quantum compact metric space to the dual of a Hilbert module with a D-norm.

Definition 3.3.1 ([47, Notation 3.8])**.** If $(\mathscr{M}, \mathsf{TN}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a metrical C*-correspondence, then for any $\mathbb{C}$-valued continuous linear functional $\varphi, \psi \in \mathscr{M}^*$, we set:

$$\mathsf{mk}_{\mathsf{TN}}(\varphi, \psi) := \sup \left\{ |\varphi(\xi) - \psi(\xi)| : \xi \in \mathrm{dom}(\mathsf{TN}), \mathsf{TN}(\xi) \leqslant 1 \right\}.$$

Since the unit ball of D-norm TN is compact, a standard argument shows that $\mathsf{mk}_{\mathsf{TN}}$ thus defined induces the weak* topology on bounded subsets of the dual $\mathscr{M}^*$ of $\mathscr{M}$.

Remark 3.3.2. The metric of Definition (3.3.1) is denoted by $\mathsf{mk}^{\mathrm{alt}}_{\mathsf{TN}}$ in [47].

We then extend, in the simplest way, the Monge-Kantorovich metric of Definition (3.3.1) on the $\mathbb{C}$-dual of a module, to a distance between families of continuous linear functionals indexed by a fixed set. We only take the distance between families of functionals indexed by the same set.

Definition 3.3.3. Let $(\mathscr{M}, \mathsf{TN}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be a metrical C*-correspondence. Let J be a nonempty set. For any two families $(\varphi_j)_{j\in J}, (\psi_j)_{j\in J} \in (\mathscr{M}^*)^J$ of continuous $\mathbb{C}$-linear functionals of $\mathbb{M}$, we set:

$$\mathsf{MK}_{\mathsf{TN}}((\varphi_j)_{j\in J}, (\psi_j)_{j\in J}) := \sup_{j\in J} \mathsf{mk}_{\mathsf{TN}}(\varphi_j, \psi_j).$$

It is immediate to check that $\mathsf{MK}_{\mathsf{TN}}$ restricts to a metric on $(\mathscr{M}^*)^J$ for any fixed nonempty set J. Its topology is stronger than the product of the weak* topology, and equal to it when J is finite, though this will not be of prime concern here.

The propinquity is defined in terms of state spaces, and for the present work, we will use the notion of *pseudo-states* for a metrical C*-correspondence, as a natural generalization of states of a quantum compact metric space. As seen in [47, Proposition 3.11], the following set is weak* compact (though not convex), and the weak* topology is metrized by the Monge-Kantorovich metric from Definition (3.3.1).

Definition 3.3.4. If $\mathbb{M} := (\mathscr{M}, \mathsf{TN}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a metrical quantum vector bundle, then a continuous linear functional $\varphi \in \mathscr{M}^*$ is a *pseudo-state* of $\mathbb{M}$ when there exist $\mu \in \mathscr{S}(\mathfrak{B})$ and $\omega \in \mathscr{M}$ with $\mathsf{TN}(\omega) \leqslant 1$ such that

$$\varphi : \xi \in \mathscr{M} \longmapsto \mu\left(\langle \xi, \omega \rangle_{\mathscr{M}}\right).$$

The set of all pseudo-states of $\mathbb{M}$ is denoted by $\widetilde{\mathscr{S}}(\mathbb{M})$.

Now, if we want to assign a distance between any two families $(a_j)_{j\in J}$ and $(b_j)_{j\in J}$ of operators on a Hilbert module $\mathscr{M}$, then a natural choice would be $\sup_{j\in J} \left|\left|\left| a_j - b_j \right|\right|\right|_{\mathscr{M}}$; however, this choice does not generalize well if we consider, instead, families of operators on different modules. Inspired from our previous work, another idea opens us when working with families of operators on a metrical quantum vector bundle $\mathbb{M}$, with D-norm TN. In this setup, we can take the Hausdorff distance between the sets $\{(\varphi \circ a_j)_{j\in J} : \varphi \in \widetilde{\mathscr{S}}(\mathbb{M})\}$ and $\{(\psi \circ b_j)_{j\in J} : \psi \in \widetilde{\mathscr{S}}(\mathbb{M})\}$ for the distance $\mathsf{MK}_{\mathsf{TN}}$ of Definition (3.3.3). The benefit of this approach is that it generalizes as follows to families of operators acting on different metrical quantum vector bundles.

Definition 3.3.5. Let $\mathbb{A}$ and $\mathbb{B}$ be two metrical quantum vector bundles. Let

$$\tau : \mathbb{A} \xleftarrow{(\Pi_{\mathbb{A}}, \pi_{\mathbb{A}}, \theta_{\mathbb{A}})} \mathbb{P} \xrightarrow{(\Pi_{\mathbb{B}}, \pi_{\mathbb{B}}, \theta_{\mathbb{B}})} \mathbb{B}$$

be a metrical tunnel from $\mathbb{A}$ to $\mathbb{B}$. Let TN be the D-norm of the metrical C*-correspondence $\mathbb{P}$.

Let J be a nonempty set. If $A := (a_j)_{j\in J}$ is a family of operators of $\mathbb{A}$, and $B := (b_j)_{j\in J}$ is a family of operators on $\mathbb{B}$, then we define the *separation* of A and B according to τ by:

$$\begin{aligned}\mathsf{sep}(A, B|\tau) := \mathsf{Haus}\left[\mathsf{MK}_{\mathsf{TN}}\right]\Big(&\left\{(\varphi \circ a_j \circ \Pi_{\mathbb{A}})_{j\in J} : \varphi \in \widetilde{\mathscr{S}}(\mathbb{A})\right\}, \\ &\left\{(\psi \circ b_j \circ \Pi_{\mathbb{B}})_{j\in J} : \psi \in \widetilde{\mathscr{S}}(\mathbb{B})\right\}\Big).\end{aligned}$$

The *dispersion* of A and B according to τ is

$$\mathsf{dis}(A, B|\tau) := \max\{\chi(\tau), \mathsf{sep}(A, B|\tau)\}.$$

We therefore obtain a natural way to discuss the convergence of families of adjoinable operators on metrical C*-correspondence, in the spirit of the Gromov-Hausdorff distance and the propinquity.

Definition 3.3.6. Let $\mathbb{A}$ and $\mathbb{B}$ be two metrical quantum vector bundles. Let J be a nonempty set. If $A := (a_j)_{j\in J}$ is a family of operators on $\mathbb{A}$, and $B := (b_j)_{j\in J}$ is a family of operators on $\mathbb{B}$, then we define the *operational propinquity* between these families as:

$$\Lambda^{\mathsf{op}}(A, B) := \inf\{\mathsf{dis}(A, B|\tau) : \tau \text{ is a tunnel from } \mathbb{A} \text{ to } \mathbb{B}\}.$$

The operational propinquity is certainly a pseudo-metric on families of operators indexed by a fixed index set, and under mild conditions, it is in fact a metric up to a natural equivalence relation, as seen in the following theorem, whose proof is a direct application of the techniques in [47].

Theorem 3.3.7. *The operational propinquity Λ^{op} is a pseudo-metric on families of operators indexed by the same set. Moreover, if we fix an index set J and we fix some $j_0 \in J$, and if we set $\mathscr{F}(J)$ to be the class:*

$$\left\{(a_j)_{j\in J} \in \mathscr{B}(\mathbb{A}) : \mathbb{A} \text{ metrical quantum vector bundle, } a_{j_0} = \mathrm{id}_{\mathfrak{A}}\right\}$$

then the restriction of Λ^{op} to $\mathscr{F}(J)$ is a metric up to the following equivalence relation. For any metrical quantum vector bundles $\mathbb{A}$ and $\mathbb{B}$, and for any family $A := (a_j)_{j\in J}$ of operators of $\mathbb{A}$, and any family $B := (b_j)_{j\in J}$ of operators of $\mathbb{B}$, both families A and B being indexed by J, we have $\Lambda^{\mathsf{op}}(A,B) = 0$ if, and only if, there exists a full metrical quantum isometry (Π,π,θ) from $\mathbb{A}$ to $\mathbb{B}$ such that, for all $j \in J$, we have $\Pi\circ a_j = b_j \circ \Pi$.

Proof. The proof follows exactly [43, Theorem 3.23], with the simplification that we only need the same indices. ☐

Our focus will be on metrical C*-correspondences constructed from metric spectral triples. If $(\mathfrak{A}_1, \mathscr{H}_1, D_1)$ is a metric spectral triple, then $\widetilde{\mathscr{S}}(\mathrm{metCor}(\mathfrak{A}_1, \mathscr{H}_1, D_1))$ is given by definition as the set of maps of the form $\varphi(\langle\cdot,\xi\rangle_{\mathscr{H}_1})$ where φ is a state of $\mathbb{C}$ — so it is is the identity on $\mathbb{C}$ — and $\xi \in \mathrm{dom}(D_1)$ with $\mathsf{DN}_1(\xi) = \|\xi\|_{\mathscr{H}} + \|D_1\xi\|_{\mathscr{H}_1} \leqslant 1$. Let now $(\mathfrak{A}_2, \mathscr{H}_2, D_2)$ be another metric spectral triple, and set DN_2 to the graph norm of D_2. Let $A := (a_j)_{j\in J}$ be a family of operators on $\mathscr{H}_1$, and let $B := (b_j)_{j\in J}$ be a family of operators on $\mathscr{H}_2$. If

$$\tau : (\mathfrak{A}_1, \mathscr{H}_1, D_1) \xleftarrow{(\Pi_1,\pi_1,\theta_1)} \mathbb{P} \xrightarrow{(\Pi_2,\pi_2,\theta_2)} (\mathfrak{A}_2, \mathscr{H}_2, D_2)$$

is some tunnel from $(\mathfrak{A}_1, \mathscr{H}_1, D_1)$ to $(\mathfrak{A}_2, \mathscr{H}_2, D_2)$, where the D-norm of $\mathbb{P}$ is denoted by TN, then the separation between A and B becomes:

$$\begin{aligned}\mathrm{sep}(A,B|\tau) = \mathsf{Haus}\left[\mathsf{MK}_{\mathsf{TN}}\right]\Big(&\left\{(\langle a_j^*\xi, \Pi_1(\cdot)\rangle)_{j\in J} : \xi \in \mathrm{dom}(\mathsf{DN}_{D}), \mathsf{DN}_{D}(\xi) \leqslant 1\right\},\\ &\left\{(\langle b_j^*\xi, \Pi_2(\cdot)\rangle)_{j\in J} : \xi \in \mathrm{dom}\left(\mathsf{DN}_{D[S]}\right), \mathsf{DN}_{D[S]}(\xi) \leqslant 1\right\}\Big).\end{aligned}$$

This expression can be unwound to give that $\mathrm{sep}(A,B|\tau)$ is the maximum of

$$\sup_{\substack{\xi\in\mathrm{dom}(D_1)\\ \mathsf{DN}_1(\xi)\leqslant 1}} \inf_{\substack{\eta\in\mathrm{dom}(D_2)\\ \mathsf{DN}_2(\eta)\leqslant 1}} \sup_{j\in J} \sup_{\mathsf{TN}(\omega)\leqslant 1} \left| \langle a_j^*\xi, \Pi_1(\omega)\rangle_{\mathscr{H}_1} - \langle b_j^*\eta, \Pi_2(\omega)\rangle_{\mathscr{H}_2} \right|$$

and

$$\sup_{\substack{\xi\in\mathrm{dom}(D_2)\\ \mathsf{DN}_2(\xi)\leqslant 1}} \inf_{\substack{\eta\in\mathrm{dom}(D_1)\\ \mathsf{DN}_1(\eta)\leqslant 1}} \sup_{j\in J} \sup_{\mathsf{TN}(\omega)\leqslant 1} \left| \langle a_j^*\eta, \Pi_1(\omega)\rangle_{\mathscr{H}_1} - \langle b_j^*\xi, \Pi_2(\omega)\rangle_{\mathscr{H}_2} \right|.$$

The spectral propinquity is defined using the above notion of dispersion for certain tunnels, where the families of operators involved arise from the natural one-parameter groups of unitaries induced by spectral triples. Intuitively, we ask for the quantum dynamics induced by spectral triples to be close for our metric if they remain close to each other for as long as possible. This construction is a special case of the covariant propinquity [43, 40, 47].

Definition 3.3.8. The *spectral propinquity* $\Lambda^{\mathsf{spec}}((\mathfrak{A}_1,\mathscr{H}_1,D\!\!\!\!/_1),(\mathfrak{A}_2,\mathscr{H}_2,D\!\!\!\!/_2))$ is defined as:

$$\inf\left\{\frac{\sqrt{2}}{2},\varepsilon>0:\exists\text{ tunnel from }(\mathfrak{A}_1,\mathscr{H}_1,D\!\!\!\!/_1)\text{ to }(\mathfrak{A}_2,\mathscr{H}_2,D\!\!\!\!/_2)\right.$$
$$\left.\mathsf{dis}\left((U_1^t)_{0\leqslant t\leqslant\frac{1}{\varepsilon}},(U_2^t)_{0\leqslant t\leqslant\frac{1}{\varepsilon}}\middle|\tau\right)<\varepsilon\right\}.$$

Originally [47], the definition involved the notion of a local almost isometric isomorphism, as introduced in [43], which, in general, allows one to define a distance between actions of different monoids. However, as seen in [33], one may simplify the definition of the spectral propinquity, thanks to to facts: the monoid in question is always $[0,\infty)$, and the action is by unitary. We also note that the number $\frac{\sqrt{2}}{2}$ in Definition (3.3.8) is needed to establish the triangle inequality (see [43, Lemma 2.11, Theorem 3.14]).

The spectral propinquity has some strong properties, in the sense that it preserves certain important structures associated with spectral triples. Most importantly, the spectrum of the Dirac operators of metric spectral triples is continuous with respect to the spectral propinquity, as a consequence of the continuity of the bounded continuous functional calculus (we note that in contrast, the Borel functional calculus is not continuous). More specifically, we proved the following.

Theorem 3.3.9 ([33])**.** *If $(\mathfrak{A}_n,\mathscr{H}_n,D\!\!\!\!/_n)_{n\in\mathbb{N}}$ is a sequence of metric spectral triples, converging to a metric spectral triple $(\mathfrak{A}_\infty,\mathscr{H}_\infty,D\!\!\!\!/_\infty)$ for the spectral propinquity, and if $f\in C_b(\mathbb{R})$ is a bounded, $\mathbb{C}$-valued continuous function over $\mathbb{R}$, then*

$$\lim_{n\to\infty}\Lambda^{\mathsf{op}}(f(D\!\!\!\!/_n),f(D\!\!\!\!/_\infty))=0.$$

A corollary of Theorem (3.3.9) is the continuity of the spectrum.

Theorem 3.3.10 ([33])**.** *If the sequence $(\mathfrak{A}_n,\mathscr{H}_n,D\!\!\!\!/_n)_{n\in\mathbb{N}}$ of metric spectral triples converges to the metric spectral triple $(\mathfrak{A}_\infty,\mathscr{H}_\infty,D\!\!\!\!/_\infty)$ for the spectral propinquity, then*

$$\mathrm{Sp}\,(D\!\!\!\!/_\infty)=\left\{\lambda\in\mathbb{R}:\exists(\lambda_n)_{n\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}\quad\forall n\in\mathbb{N}\quad\lambda_n\in\mathrm{Sp}\,(D\!\!\!\!/_n)\ \textit{and}\ \lambda=\lim_{n\to\infty}\lambda_n\right\}.$$

The next natural question is whether the mutliplicities of the eigenvalues of Dirac operators converge. Of course, in general, eigenvalues could "merge" at the limit, so the most general result is as follows.

Theorem 3.3.11 ([33])**.** *If $(\mathfrak{A}_n,\mathscr{H}_n,D\!\!\!\!/_n)_{n\in\mathbb{N}}$ is a sequence of metric spectral triples which converges to a metric spectral triple $(\mathfrak{A}_\infty,\mathscr{H}_\infty,D\!\!\!\!/_\infty)$ for the spectral propinquity,* *if* $\lambda \in \mathrm{Sp}(\mathbb{D}_\infty)$*, and if there exists* $\delta > 0$ *and* $N \in \mathbb{N}$ *such that* $(\lambda-\delta, \lambda+\delta) \cap \mathrm{Sp}(\mathbb{D}_n)$ *is a singleton* $\{\lambda_n\}$ *for all* $n \geqslant N$*, then we assert:*

$$\liminf_{n\to\infty} \mathrm{multiplicity}(\lambda_n|\mathbb{D}_n) \geqslant \mathrm{multiplicity}(\lambda|\mathbb{D}_\infty).$$

To get a stronger form of continuity for mutliplicities, we offer a sufficient condition, which restricts the potential growth of the eigenvalues.

Theorem 3.3.12 ([33])**.** *If* $(\mathfrak{A}_n, \mathscr{H}_n, \mathbb{D}_n)_{n\in\mathbb{N}}$ *is a sequence of metric spectral triples converging, for the spectral propinquity, to a metric spectral triple* $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \mathbb{D}_\infty)$*, and*

1. *if* $\lambda \in \mathrm{Sp}(\mathbb{D}_\infty)$*,*
2. *there exists* $\delta > 0$ *and* $N \in \mathbb{N}$ *such that, for all* $n \geqslant N$*, the intersection* $\mathrm{Sp}(\mathbb{D}_n) \cap (\lambda-\delta, \lambda+\delta)$ *is a singleton, denoted by* $\{\lambda_n\}$*,*
3. *if* $(\mathrm{multiplicity}(\lambda_n|\mathbb{D}_n))_{n\in\mathbb{N}}$ *converges in* $\mathbb{N}$ *— i.e., is eventually constant,*

then

$$\lim_{n\to\infty} \mathrm{multiplicity}(\lambda_n|\mathbb{D}_n) = \mathrm{multiplicity}(\lambda|\mathbb{D}_\infty).$$

We now then use Theorem (3.3.10) to compute the spectrum of operators as limits of finite dimensional spectral triples [33], and we can apply Theorems (3.3.9) and (3.3.10) to derive continuity results for the spectral actions, for instance.

3.4 EXAMPLES: INDUCTIVE LIMITS OF GROUPS

We then turn to the more specific context of inductive sequences of metric spectral triples. Inductive sequences of spectral triples were introduced in [18], and are a natural source of spectral triples; our interest is in the convergence of such sequences for the spectral propinquity, i.e. in the sense of an actual metric.

We begin by recalling the following notion of inductive limit for spectral triples.

Definition 3.4.1. Let $\mathfrak{A}_\infty = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right)$ be a C*-algebra which is the closure of an increasing sequence of C*-subalgebras $(\mathfrak{A}_n)_{n\in\mathbb{N}}$ in $\mathfrak{A}_\infty$, with the unit of $\mathfrak{A}_\infty$ in $\mathfrak{A}_0$. A spectral triple $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \mathbb{D}_\infty)$ is the inductive limit of a sequence $(\mathfrak{A}_n, \mathscr{H}_n, \mathbb{D}_n)_{n\in\mathbb{N}}$ of spectral triples when:

1. $\mathscr{H}_\infty = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}}\right)\mathscr{H}_n$, where each $\mathscr{H}_n$ is a Hilbert subspace of $\mathscr{H}_\infty$,
2. for each $n \in \mathbb{N}$, the restriction of $\mathbb{D}_\infty$ to $\mathrm{dom}(\mathbb{D}_n)$ is $\mathbb{D}_n$,
3. for each $n \in \mathbb{N}$, the subspace $\mathscr{H}_n$ is reducing for $\mathfrak{A}_n$, which is equivalent to $\mathfrak{A}_n\mathscr{H}_n \subseteq \mathscr{H}_n$.

We note, using the notation of Definition (3.4.1), that the operator which, to any $\xi \in \bigcup_{n\in\mathbb{N}} \mathrm{dom}(\mathbb{D}_n)$, associates $\mathbb{D}_n\xi$ whenever $\xi \in \mathrm{dom}(\mathbb{D}_n)$ for any $n \in \mathbb{N}$, is indeed well-defined, and shown in [18] to be essentially self-adjoint, so $\mathbb{D}_\infty$ is the closure of this operator.

For our purpose, the following result from [18] show some consequences of a spectral triple being an inductive limit.

Theorem 3.4.2 ([18, Theorem 3.1, partial])**.** *If $(\mathfrak{A}_n, \mathscr{H}_n, \mathbb{D}_n)_{n\in\mathbb{N}}$ is an inductive sequence of spectral triples converging to a* spectral triple $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \mathbb{D}_\infty)$*, then for any $\mathbb{C}$-valued continuous function $f \in C_0(\mathbb{R})$ which vanishes at infinity, the sequence $(P_n f(\mathbb{D}_n) P_n)_{n\in\mathbb{N}}$ converges to $f(\mathbb{D}_\infty)$ in norm.*

From a metric perspective, inductive limits of spectral triples may provide interesting examples of convergence for the spectral propinquity. In turn, the properties of the spectral propinquity, including the continuity of the spectrum and others, are tranferred to this setup of inductive limits, strengthening Theorem (3.4.2).

Theorem 3.4.3. *Let $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \mathbb{D}_\infty)$ be a metric spectral triple which is the inductive limit of a sequence $(\mathfrak{A}_n, \mathscr{H}_n, \mathbb{D}_n)_{n\in\mathbb{N}}$ of metric spectral triples, in the sense of Definition (3.4.1). For each $n \in \mathbb{N}\cup\{\infty\}$, let*

$$\mathrm{dom}(\mathsf{L}_n) := \left\{a \in \mathfrak{A}_n : a = a^*, a\,\mathrm{dom}(\mathbb{D}_n) \subseteq \mathrm{dom}(\mathbb{D}_n) \text{ and } [\mathbb{D}_n, a] \text{ is bounded}\right\},$$

and, for all $a \in \mathrm{dom}(\mathsf{L}_n)$, let $\mathsf{L}_n(a)$ be the operator norm of $[\mathbb{D}_n, a]$.

If there exists a bridge builder $\pi : (\mathfrak{A}_\infty, \mathsf{L}_\infty) \to (\mathfrak{A}_\infty, \mathsf{L}_\infty)$ for $((\mathfrak{A}_n, \mathsf{L}_n)_{n\in\mathbb{N}}, (\mathfrak{A}_\infty, \mathsf{L}_\infty))$ which is a full quantum isometry of $(\mathfrak{A}_\infty, \mathsf{L}_\infty)$, i.e. such that $\pi(\mathrm{dom}(\mathsf{L}_\infty)) \subseteq \mathrm{dom}(\mathsf{L}_\infty)$ and $\mathsf{L}_\infty \circ \pi = \mathsf{L}_\infty$ on $\mathrm{dom}(\mathsf{L}_\infty)$, then

$$\lim_{n\to\infty} \Lambda^{\mathsf{spec}}((\mathfrak{A}_n, \mathscr{H}_n, \mathbb{D}_n), (\mathfrak{A}_\infty, \mathscr{H}_\infty, \mathbb{D}_\infty)) = 0.$$

Theorem 3.4.4. *Let $G = \bigcup_{n\in\mathbb{N}} G_n$ be an Abelian discrete group, with $(G_n)_{n\in\mathbb{N}}$ a strictly increasing sequence of subgroups of G. Let σ be a 2-cocycle of G, with values in $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.*

Let $\mathbb{L}_H$ be a length function over G whose restriction to G_n is proper for all $n \in \mathbb{N}$, such that the sequence $(G_n)_{n\in\mathbb{N}}$ converges to G for the Hausdorff distance induced on the closed subsets of G by $\mathbb{L}_H$. Let

$$\mathbb{F} : g \in G \longmapsto \mathrm{scale}(\min\{n \in \mathbb{N} : g \in G_n\}),$$

where $\mathrm{scale} : \mathbb{N} \to [0, \infty)$ *is a strictly increasing function.*

If the proper length function $\mathbb{L} := \max\{\mathbb{L}_H, \mathbb{F}\}$ satisfies that, for some $\theta > 1$, there exists $c > 0$ such that for all $r \geqslant 1$:

$$\left|\left\{g \in G : \mathbb{L}(g) \leqslant \theta \cdot r\right\}\right| \leqslant c \left|\left\{g \in G : \mathbb{L}(g) \leqslant r\right\}\right|,$$

then

$$\lim_{n\to\infty} \Lambda^{\mathsf{spec}}((C^*(G,\sigma), \ell^2(G) \otimes \mathbb{C}^2, \mathbb{D}), (C^*(G_n,\sigma), \ell^2(G_n) \otimes \mathbb{C}^2, \mathbb{D}_n)) = 0,$$

where for all $n \in \mathbb{N} \cup \{\infty\}$ and for all (ξ_1, ξ_2) in

$$\left\{ \xi \in \ell^2(G_n) \otimes \mathbb{C}^2 : \sum_{g \in G_n} (\mathbb{L}_H(g)^2 + \mathbb{F}(g)^2) \left\| \xi(g) \right\|_{\mathbb{C}^2}^2 < \infty \right\},$$

we set

$$D\!\!\!/\xi : g \in G \longmapsto \begin{pmatrix} \mathbb{F}(g)\xi_2(g) + \mathbb{L}_H(g)\xi_1(g) \\ \mathbb{F}(g)\xi_2(g) - \mathbb{L}_H(g)\xi_1(g) \end{pmatrix}.$$

In the above spectral triples, $C^*(G,\sigma)$ and $C^*(G_n,\sigma)$ act via their left regular σ-projective representations.

Corollary 3.4.5. *Fix a prime number $p \in \mathbb{N}$ and $d \in \mathbb{N} \setminus \{0,1\}$. For each $n \in \mathbb{N}$, let*

$$G_n := \left(\frac{1}{p^n} \mathbb{Z} \right)^d$$

and

$$G_\infty := \left(\mathbb{Z}\left[\frac{1}{p} \right] \right)^d.$$

Fix a 2-cocycle σ on G_∞ such that $\forall g \in G_\infty \quad \sigma(g,-g) = 1$.

Let $\mathbb{L}_H$ be the restriction to G_∞ of some norm on $\mathbb{R}^2$. We define $\mathbb{F}$ by setting, for all $g \in G_\infty$:

$$\mathbb{F}(g) := \min \left\{ p^n : g \in \left(\frac{1}{p^n} \right)^d \right\}.$$

Let E be an even dimensional hermitian space, with γ_1, γ_2 be two unitaries on E such that, for all $j,k \in \{1,2\}$:

$$\gamma_j \gamma_k + \gamma_k \gamma_j = \begin{cases} 2 \text{ if } j = k, \\ 0 \text{ otherwise.} \end{cases}$$

If we define, for all $n \in \overline{\mathbb{N}}$, the operator

$$D\!\!\!/_n := M_{\mathbb{L}_H} \otimes \gamma_1 + M_{\mathbb{F}} \otimes \gamma_2 \text{ on } \mathrm{dom}(D\!\!\!/_n)$$

on the domain

$$\mathrm{dom}(D\!\!\!/_n) := \left\{ \xi \in \ell^2(G_n, E) : \sum_{g \in G_n} (\mathbb{L}_H(g)^2 + \mathbb{F}(g)^2) \left\| \xi(g) \right\|_E^2 < \infty \right\},$$

then, for all $n \in \overline{\mathbb{N}}$, the triple $(C^*(G_n,\sigma), \ell^2(G_n,E), D\!\!\!/_n)$ is a metric spectral triple, and:

$$\lim_{n\to\infty} \Lambda^{\mathsf{spec}}((C^*(G_n,\sigma), \ell^2(G_n,E), D\!\!\!/_n), (C^*(G_\infty,\sigma), \ell^2(G_\infty,E), D\!\!\!/_\infty)) = 0.$$

Moreover, for each $n \in \mathbb{N}$, the sequence $(C^*(G_n,\sigma), \mathsf{L}_k)_{k \geqslant n}$ of quantum compact metric spaces converge to $(C^*(G_n,\sigma), \mathsf{L}_\infty)$ in the Lipschitz distance.

Corollary 3.4.6. *Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a sequence of nonzero natural numbers such that $\left(\frac{\alpha_{n+1}}{\alpha_n}\right)_{n\in\mathbb{N}}$ is a bounded sequence of prime numbers, and let*

$$\mathbb{Z}(\alpha) := \left\{\zeta \in \mathbb{C} : \exists n \in \mathbb{N} \quad \zeta^{\alpha_n} = 1\right\}.$$

Define:

$$G_\infty := \mathbb{Z}(\alpha) \times \mathbb{Z} \text{ and } \forall n \in \mathbb{N} \quad G_n := \widehat{\mathbb{Z}/_{\alpha_n}} \times \mathbb{Z},$$

i.e. $G_n = \{(\zeta, z) \in G_\infty : z \in \mathbb{Z}, \zeta^{\alpha_n} = 1\}$. Let σ be a 2-cocycle of G_∞.

Let $\mathbb{L}_Z$ be the restriction of any continuous length function on $\mathbb{T}$ to $\mathbb{Z}(\alpha)$, and define $\mathbb{L}_H : (u,z) \in G_\infty \mapsto \mathbb{L}_Z(u) + |z|$.

For all $\zeta \in \mathbb{Z}(\alpha)$, set:

$$\mathbb{F}(\zeta) := \min\{n \in \mathbb{N} : u^n = 1\}.$$

Let E be a Hermitian vector space, and let γ_1,γ_2 be unitaries such that $\gamma_1\gamma_2 = -\gamma_2\gamma_1$ and $\gamma_1^2 = \gamma_2^2 = 1_E$.

If we set, for all $n \in \overline{\mathbb{N}}$,

$$\not{D}_n := M_{\mathbb{L}_H} \otimes \gamma_1 + M_{\mathbb{F}} \otimes \gamma_2,$$

then for all $n \in \mathbb{N}$, the spectral triple $(C^*(G_n,\sigma), \ell^2(G_n) \otimes E, \not{D}_n)$ is metric, and

$$\lim_{n\to\infty} \Lambda^{\mathsf{spec}}\left(\left(C^*(G_n,\sigma), \ell^2(G_n)\otimes E, \not{D}_n\right), \left(C^*(\mathbb{Z}(\alpha)\times\mathbb{Z},\sigma), \ell^2(\mathbb{Z}(\alpha)\times\mathbb{Z})\otimes E, \not{D}_\infty\right)\right) = 0.$$

3.5 AF ALGEBRAS

Antonescu/Ivan and Christensen constructed in [5] a metric spectral triple on AF algebras. We now apply some of our results to prove the convergence of isometry groups of their spectral triple. We begin with a description of the setup of [5].

Let $\mathfrak{A} = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right)$ be a unital C*-algebra arising as the closure of the union of an increasing union of finite dimensional C*-subalgebras — i.e., $\mathfrak{A}$ is an AF algebra. We assume for convenience that $\mathfrak{A}_0 = \mathbb{C}1$ where 1 is the unit of $\mathfrak{A}$. Let $\varphi \in \mathscr{S}(\mathfrak{A})$ be a *faithful* state of $\mathfrak{A}$, and let $L^2(\mathfrak{A},\varphi)$ be the GNS Hilbert space obtained by completion of $\mathfrak{A}$ for the inner product $a, b \in \mathfrak{A} \mapsto \varphi(b^* a)$. Of course, $\mathfrak{A}$ acts by left multiplication on $L^2(\mathfrak{A},\varphi)$. We write Ω for the unit 1 of $\mathfrak{A}$ when seen as a vector in $L^2(\mathfrak{A},\varphi)$, which is then a cyclic and separating vector for $\mathfrak{A}$.

For each $n \in \mathbb{N}$, we identify $L^2(\mathfrak{A}_n,\varphi)$ with the closure of $\mathfrak{A}_n\Omega$ in $L^2(\mathfrak{A},\varphi)$, and we denote the orthogonal projection of $L^2(\mathfrak{A},\varphi)$ onto $L^2(\mathfrak{A}_n,\varphi)$ by P_n. For all $a \in \mathfrak{A}$, let $\mathbb{E}_n(a)$ be defined by: $\forall a \in \mathfrak{A}_n \quad \mathbb{E}_n(a)\Omega := P_n(a\Omega)$ — this indeed is well-defined since Ω is separating. Since $\mathfrak{A}_n$ is finite dimensional, it agrees, as a vector space, with $L^2(\mathfrak{A}_n,\varphi)$, and thus $\mathbb{E}_n(a) \in \mathfrak{A}_n$. It is easy to check that $\mathbb{E}_n$ thus defined is a continuous linear function. In fact, when φ is also a trace, then $\mathbb{E}_n$ is the unique conditional expectation from $\mathfrak{A}_\infty$ onto $\mathfrak{A}_n$ such that $\varphi \circ \mathbb{E}_n = \varphi$.

We now follow the construction of a spectral triple on AF algebras given in [5]. For each $n \in \mathbb{N}\setminus\{0\}$, we now let $Q_n := P_n - P_{n-1}$; of course Q_n is a projection since P_n and

P_{n-1} commute and by construction, $P_n \geqslant P_{n-1}$. Now fix $a \in \mathfrak{A}_n$, so $a\Omega \in L^2(\mathfrak{A}_n, \varphi)$. By construction, $P_m(a\Omega) = a\Omega$, and thus $Q_m(a\Omega) = 0$ for all $m > n$. By continuity of Q_m, we thus conclude that $Q_m L^2(\mathfrak{A}_n, \varphi) = \{0\}$ for all $m > n$. This allows us to define the following operator.

For any strictly increasing sequence $\lambda := (\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers, we define

$$\forall \xi \in \bigcup_{n\in\mathbb{N}} L^2(\mathfrak{A}_n, \varphi) \quad D_\lambda \xi := \sum_{n=0}^{\infty} \lambda_n Q_n \text{ on } \bigcup_{n\in\mathbb{N}} L^2(\mathfrak{A}_n, \varphi), \tag{3.5.1}$$

noting that since $(L^2(\mathfrak{A}_n, \varphi))_{n\in\mathbb{N}}$ is increasing for inclusion, the set $\bigcup_{n\in\mathbb{N}} L^2(\mathfrak{A}_n, \varphi)$ is a subspace of $L^2(\mathfrak{A}, \varphi)$. Since $\bigcup_{n\in\mathbb{N}} L^2(\mathfrak{A}_n, \varphi)$ is dense in $L^2(\mathfrak{A}, \varphi)$, the operator D_λ is defined on a dense subspace. It is obviously symmetric, by construction, and in fact, if

$$T = \sum_{n=1}^{\infty} ((\lambda_n + i)^{-1}) Q_n$$

where the series converge in norm, then T is bounded and $T(D_\lambda + i)\xi = \xi$ for all $\xi \in \bigcup_{n\in\mathbb{N}} L^2(\mathfrak{A}_n, \varphi)$. Therefore, D_λ is essentially self-adjoint. Let $\mathbb{D}_\lambda$ be the closure of D_λ. It is then easy to see that $\mathbb{D}_\lambda$ is a self-adjoint operator and T is the inverse of $\mathbb{D}_\lambda + i$. Since T is compact, $\mathbb{D}_\lambda$ has compact resolvent.

Now fix $a \in \mathfrak{A}_n$. We note that a commutes with P_k for $k \geqslant n$ by construction, since $ab \in \mathfrak{A}_k$ for all $b \in \mathfrak{A}_k$ with $k \geqslant n$. In turn, this shows that

$$[\mathbb{D}_\lambda, a] = \sum_{j=0}^{\infty} \lambda_j [Q_j, a] = \sum_{j=0}^{n} \lambda_j [Q_j, a] \tag{3.5.2}$$

$$= P_k \left(\sum_{j=0}^{\infty} [Q_j, a] \right) P_k = P_k [\mathbb{D}_\lambda, a] P_k. \tag{3.5.3}$$

From this, we see that $\{a \in \mathfrak{A}_\infty : a\,\mathrm{dom}(\mathbb{D}_\lambda) \subseteq \mathrm{dom}(\mathbb{D}_\lambda)\}$ is dense in $\mathfrak{A}_\infty$, as it contains $\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$, and thus $(\mathfrak{A}_\infty, L^2(\mathfrak{A}_\infty, \varphi), \mathbb{D}_\lambda)$ is a spectral triple. Moreover, so is $(\mathfrak{A}_n, L^2(\mathfrak{A}_n, \varphi), \mathbb{D}_n)$ where $\mathbb{D}_n$ is the restriction of $\mathbb{D}_\infty$ to $L^2(\mathfrak{A}_n, \varphi)$, which is now a bounded self-adjoint operator. Moreover, by Equation (3.5.2), we have for all $a \in \mathfrak{A}_n$:

$$|||[\mathbb{D}_n, a]|||_{L^2(\mathfrak{A}_n, \varphi)} = |||[\mathbb{D}_\infty, a]|||_{L^2(\mathfrak{A}_\infty, \varphi)}.$$

Not all choices of a sequence λ as above leads to a metric spectral triple. The following result addresses this matter in [5].

Theorem 3.5.1 (Theorem 2.1)**.** *There exists a strictly increasing sequence $\lambda := (\lambda_n)_{n\in\mathbb{N}}$ of positive numbers, and a summable sequence $(\beta_n)_{n\in\mathbb{N}}$, such that*

$$\forall a \in \mathfrak{sa}(\mathfrak{A}_\infty) \qquad \|\mathbb{E}_n(a) - \mathbb{E}_{n+1}(a)\|_{\mathfrak{A}} < \beta_n |||[\mathbb{D}_\lambda, a]|||_{L^2(\mathfrak{A}_\infty, \varphi)}. \tag{3.5.4}$$

Consequently, for this choice of sequence λ, the spectral triples $\left(\mathfrak{A}_n, L^2(\mathfrak{A}_n, \varphi), (\mathbb{D}_\lambda)_{|L^2(\mathfrak{A}_n, \varphi)}\right)$ are metric for all $n \in \mathbb{N} \cup \{\infty\}$.

Since we have constructed metric spectral triples, we have constructed quantum compact metric spaces and we can ask about their convergence for the propinquity. In fact, we now prove a convergence result for the *spectral propinquity* Λ^{spec} of the spectral triples, which implies the convergence of the underlying quantum compact metric spaces.

Moreover, by Theorem (), to show convergence, it is sufficient to find a bridge builder which is also a full quantum isometry.

We refer to [47] for the definition of the spectral propinquity, and we refer to Theorem (3.4.3) to see that, when working in the context of AF algebras, it is sufficient for the following result to find a bridge builder which is also a full quantum isometry.

Theorem 3.5.2. *Let* $\mathfrak{A} = \mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n\right)$ *be a unital C*-algebra arising as the union of an increasing sequence of finite dimensional C*-subalgebras* $\mathfrak{A}_n$ *with* $\mathfrak{A}_0 = \mathbb{C}1$*. Let* $\varphi \in \mathscr{S}(\mathfrak{A})$ *be a faithful state of* $\mathfrak{A}$*.*

If $\lambda := (\lambda_n)_{n\in\mathbb{N}}$ *is a strictly increasing sequence of positive real numbers such that* $(\mathfrak{A}_\infty, L^2(\mathfrak{A}_\infty,\varphi), \not{D}_\lambda)$ *is a metric spectral triple, then, setting* $\not{D}_n$ *to be the restriction of* $\not{D}_\lambda$ *to* $\mathrm{dom}(\not{D}_\lambda) \cap \mathscr{H}_n$*, we have* $(\mathfrak{A}_n, \mathscr{H}_n, \not{D}_n)$ *is a metric spectral triple, and*

$$\lim_{n\to\infty} \Lambda^{\mathsf{spec}}((\mathfrak{A}_n, L^2(\mathfrak{A}_n,\varphi), \not{D}_n), (\mathfrak{A}_\infty, L^2(\mathfrak{A}_\infty,\varphi), \not{D}_\lambda))) = 0,$$

and therefore, in particular,

$$\lim_{n\to\infty} \Lambda^*\left(\left(\mathfrak{A}_n, |||[(\not{D}_n,\cdot]|||_{L^2(\mathfrak{A}_n,\varphi)}\right), \left(\mathfrak{A}_\infty, |||[\not{D}_\lambda,\cdot]|||_{L^2(\mathfrak{A}_\infty,\varphi)}\right)\right) = 0.$$

Proof. We will prove that the identity of $\mathfrak{A}_\infty$ is a bridge builder. First, we note that for all $a \in \bigcup_{n\in\mathbb{N}} \mathfrak{sa}(\mathfrak{A}_n) \subseteq \mathrm{dom}(\mathsf{L}_\infty)$, the sequence $(\mathbb{E}_n(a))_{n\in\mathbb{N}}$ is eventually constant equal to a, and thus $\lim_{n\to\infty} \mathbb{E}_n(a) = a$. If $a \in \mathfrak{sa}(\mathfrak{A})$, then for all $\varepsilon > 0$, there exists $a' \in \bigcup_{n\in\mathbb{N}} \mathfrak{sa}(\mathfrak{A}_n)$ such that $\|a - a'\|_{\mathfrak{A}_\infty} < \frac{\varepsilon}{2}$; let $N' \in \mathbb{N}$ such that $a' \in \mathfrak{A}_n$ for all $n \geqslant N'$. Then

$$\begin{aligned}\|a - \mathbb{E}_n(a)\|_{\mathfrak{A}_\infty} &\leqslant \|a - a'\|_{\mathfrak{A}_\infty} + \|a' - \mathbb{E}_n(a')\|_{\mathfrak{A}_\infty} + \|\mathbb{E}_n(a' - a)\|_{\mathfrak{A}_\infty} \\ &< \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus $\lim_{n\to\infty} \|\mathbb{E}_n(a) - a\|_{\mathfrak{A}_\infty} = 0$ for all $a \in \mathfrak{A}$.

Now, fix $\varepsilon > 0$. Let $N \in \mathbb{N}$ such that, if $n \geqslant N$, then $\sum_{n\geqslant N} \beta_m < \varepsilon$. For all $n \in \overline{\mathbb{N}}$, let L_n be defined as $\mathsf{L}_{\not{D}_n}$.

Let $a \in \mathrm{dom}(\mathsf{L}_\infty)$. By Equation (3.5.2), we have

$$[\not{D}_n, \mathbb{E}_n(a)] = P_n[\not{D}, a]P_n,$$

so $\mathsf{L}_n(\mathbb{E}_n(a)) \leqslant \mathsf{L}_\infty(a)$. On the other hand,

$$\|a - \mathbb{E}_n(a)\|_{\mathfrak{A}} \leqslant \sum_{k=n}^{\infty} \|\mathbb{E}_{k+1}(a) - \mathbb{E}_k(a)\|_{\mathfrak{A}} \leqslant \mathsf{L}_\infty(a) \sum_{k=n}^{\infty} \beta_k < \varepsilon \mathsf{L}_\infty(a).$$

Moreover, if $a \in \mathrm{dom}(\mathsf{L}_n)$, then we simply note that $\mathsf{L}_\infty(a) = \mathsf{L}_n(a)$ and $\|a - a\|_{\mathfrak{A}_\infty} = 0$.

We have shown that the identity is a bridge builder; of course it is also a full quantum isometry. Therefore, by Theorem (3.4.3), we have the claimed convergence. □

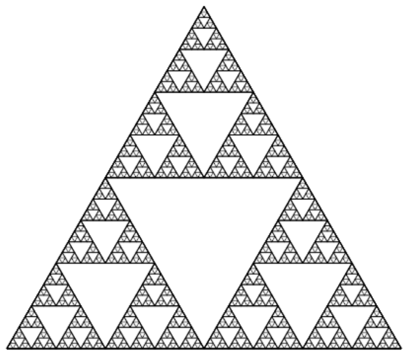

FIGURE 3.1: The Sierpiński gasket

3.6 SIERPIŃSKI GASKET

THE SIERPIŃSKI GASKET

The Sierpiński gasket $\mathscr{SG}_\infty$ is a fractal, constructed as the attractor set of an iterated function system (IFS) of affine functions of the plane. Specifically, let

$$v_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}.$$

We write $V_0 = \{v_0, v_1, v_2\}$. Let $\Delta_{0,1} = \mathscr{SG}_0$ be the boundary of the convex hull of V_0 in $\mathbb{R}^2$ — i.e., $\Delta_{0,1}$ is an equilateral triangle in the plane, whose edges have length 1. Let $L_0 = \{\Delta_{0,1}\}$.

We define three similitudes of the plane by letting for each $j \in \{0,1,2\}$,

$$T_j : x \in \mathbb{R}^2 \longmapsto \frac{1}{2}\left(x + v_j\right) \in \mathbb{R}^2.$$

We will use an explicit construction of $\mathscr{SG}_\infty$ as a limit of finite graphs in $\mathbb{R}^2$, defined inductively. For all $n \in \mathbb{N}$, $n > 0$, we set

$$L_n = \left\{\Delta_{n,j} : j \in \{1,\ldots,3^n\}\right\},$$

where

- $\Delta_{n+1,j+r3^n} = T_r \Delta_{n,j}$, for all $j \in \{1,\ldots,3^n\}$ and $r \in \{0,1,2\}$,
- $V_{n+1} = \bigcup_{r=0}^{2} T_r V_n$.

For each $n \in \mathbb{N}$, we define the set

$$\mathscr{SG}_n = \bigcup L_n = \bigcup_{j=1}^{3^n} \Delta_{n,j}.$$

We observe for later use that, by induction, for all $n \in \mathbb{N}$:

1. $\Delta_{n,j}$ is an equilateral triangle whose edges have length $\frac{1}{2^n}$;
2. the set of all vertices of the triangles in L_n is V_n;
3. if $j \in \{1,\ldots,3^n\}$, then
$$\Delta_{n,j} \subseteq \bigcup_{r=1}^{3} \Delta_{n+1,3(j-1)+r} \subseteq \overline{\mathrm{co}}\left(\Delta_{n,j}\right);$$
4. $\mathscr{S}\mathscr{G}_n \subseteq \mathscr{S}\mathscr{G}_{n+1}$;
5. if $j,k \in \{1,\ldots,3^n\}$, and $j \neq k$, then $\Delta_{n,j} \cap \Delta_{n,k}$ is empty or a singleton containing the common vertex to both triangles;
6. $\mathscr{S}\mathscr{G}_n$ is path connected.

All of these observations above follow from the fact that affine bijections preserve triangles, scale length (here, by $\frac{1}{2}$), and preserve intersections; they all can be proved by induction.

Our construction implies the following key metric property:

$$\forall n,m \in \mathbb{N} \quad m \geqslant n \implies \mathsf{Haus}\left[\|\cdot\|_{\mathbb{R}^2}\right](\mathscr{S}\mathscr{G}_m, V_n) \leqslant \frac{1}{2^n}.$$

We now define the *Sierpiński gasket*, using the notation of this section.

Definition 3.6.1. The Sierpiński gasket $\mathscr{S}\mathscr{G}_\infty$ is the closure of $\bigcup_{n\in\mathbb{N}} \mathscr{S}\mathscr{G}_n$.

We further define $V_\infty = \bigcup_{n\in\mathbb{N}} V_n$.

By construction, the Sierpiński gasket is compact. It can also easily be checked that $\mathscr{S}\mathscr{G}_\infty$ is invariant under the map

$$X \subseteq \mathbb{R}^2 \mapsto \bigcup_{r=0}^{2} T_r X;$$

so it is indeed the attractor of the iterated functions system (T_0, T_1, T_2) and is, in fact, a self-similar set; namely,

$$\mathscr{S}\mathscr{G}_\infty = \bigcup_{r=0}^{2} T_r \mathscr{S}\mathscr{G}_\infty.$$

It is immediate from our construction that

$$\forall n \in \mathbb{N}, \quad \mathsf{Haus}\left[\|\cdot\|_{\mathbb{R}^2}\right](\mathscr{S}\mathscr{G}_\infty, V_n) \leqslant \frac{1}{2^n}.$$

The set V_∞ is therefore dense in $\mathscr{S}\mathscr{G}_\infty$. We also note that, by construction,

$$\forall n \in \mathbb{N}, \quad \mathsf{Haus}\left[\|\cdot\|_{\mathbb{R}^2}\right](\mathscr{S}\mathscr{G}_n, \mathscr{S}\mathscr{G}_\infty) \leqslant \frac{1}{2^n}; \tag{3.6.1}$$

hence, $\mathscr{SG}_\infty$ is the limit of $(\mathscr{SG}_n)_{n\in\mathbb{N}}$ for $\mathsf{Haus}\left[\|\cdot\|_{\mathbb{R}^2}\right]$.

However, for each $n \in \overline{\mathbb{N}}$, we will work with the intrinsic metric on the set $\mathscr{SG}_n$, which is different from the restriction to $\mathscr{SG}_n$ of the metric of $\mathbb{R}^2$. Indeed, since for each $n \in \overline{\mathbb{N}}$, the set $\mathscr{SG}_n$ is path connected, we can define for any two $x, y \in \mathscr{SG}_n$,

$$d_n(x,y) = \inf\left\{\text{length}(\gamma) : \gamma : [0,1] \to \mathscr{SG}_n, \gamma(0) = x, \gamma(1) = y, \gamma \text{ continuous}\right\}.$$

Here and henceforth (see, e.g., [20]), for any natural number $p \geqslant 1$, and for any curve in a compact subset X of $\mathbb{R}^p$, i.e., a continuous map $\gamma : [0,1] \to X$, we define the *length* of γ by

$$\text{length}(\gamma) = \sup\left\{\sum_{j=0}^{k} \left\|\gamma(t_j) - \gamma(t_{j+1})\right\|_{\mathbb{R}^p} : k \in \mathbb{N}, 0 = t_0 < t_1 < \ldots < t_k = 1\right\},$$

allowing for the value ∞ for length(γ) (the curves with finite length are called *rectifiable*), and where $\|\cdot\|_{\mathbb{R}^p}$ is the Euclidean norm on $\mathbb{R}^p$.

By the Hopf–Rinow theorem for length spaces [20], and since $\mathscr{SG}_n$ is path connected and compact, there exists a continuous function $\gamma : [0,1] \to \mathscr{SG}_n$ which is a geodesic from $\gamma(0) = x$ to $\gamma(1) = y$; i.e., for all $t \leqslant t' \in [0,1]$, we have length$(\gamma|[t,t']) = d_n(\gamma(t), \gamma(t')) = \lambda|t - t'|$, where $\lambda = \text{length}(\gamma)$. This last equality shows that γ is injective. We will use these observations in several proofs below.

In general, the canonical inclusion of $\mathscr{SG}_n$ into $\mathscr{SG}_\infty$ is not an isometry from $(\mathscr{SG}_n, d_n)$ to $(\mathscr{SG}_\infty, d_\infty)$. For instance, we see that

$$d_0\left(v_2, \begin{pmatrix}\frac{1}{2}\\0\end{pmatrix}\right) = \frac{3}{2}, \text{ yet } d_\infty\left(v_2, \begin{pmatrix}\frac{1}{2}\\0\end{pmatrix}\right) = d_1\left(v_2, \begin{pmatrix}\frac{1}{2}\\0\end{pmatrix}\right) = 1.$$

This simple computation also shows that, of course, for any $n \in \overline{\mathbb{N}}$, the space $(\mathscr{SG}_n, d_n)$ is not a metric subspace of $\mathbb{R}^2$ (with its usual metric); namely, while $\mathscr{SG}_n$ is a subset of $\mathbb{R}^2$, the restriction to $\mathscr{SG}_n$ of the usual Euclidean metric on $\mathbb{R}^2$ is not equal to d_n. It is obvious, nonetheless, that

$$\forall n \in \overline{\mathbb{N}}, \quad \forall x, y \in \mathscr{SG}_n, \quad \left\|x - y\right\|_{\mathbb{R}^2} \leqslant d_\infty(x,y) \leqslant d_n(x,y).$$

However, we make the following observation, which will prove helpful later in this work.

Lemma 3.6.2. *For all $n \in \mathbb{N}$, the metrics d_n and d_∞ agree on V_n. In fact, any geodesic between two elements of V_n in $(\mathscr{SG}_\infty, d_\infty)$ is also a geodesic in $(\mathscr{SG}_n, d_n)$.*

Let CP be the unital Abelian C*-algebra of all $\mathbb{C}$-valued continuous functions f over $[-1,1]$ such that $f(-1) = f(1)$:

$$\text{CP} = \left\{f \in C([-1,1]) : f(-1) = f(1)\right\}.$$

The Gelfand spectrum of CP is, of course, homeomorphic to the unit circle in $\mathbb{C}$; we will identify it with the image $\mathbb{T}$ of $[-1,1]$ under the map

$$x \in [-1,1] \longmapsto \exp(i\pi x).$$

The Sierpiński gasket is a special case of a general type of fractals.

Definition 3.6.3. A *piecewise C^1-fractal curve* X is a compact path connected subset of $\mathbb{R}^n$ such that, for some sequence $(C_j)_{j\in\mathbb{N}}$ of rectifiable C^1-curves in $\mathbb{R}^n$ with $\lim_{j\to\infty} \text{length}(C_j) = 0$, the following assertions hold:

1. $X =$ the closure of $\bigcup_{j\in\mathbb{N}} \text{range}\left(C_j\right)$,
2. there exists a dense subset $\mathscr{B}$ of X (for the topology induced by the geodesic distance on X) consisting of endpoints of the curves in the sequence $(C_j)_{j\in\mathbb{N}}$ and such that for all $p \in \mathscr{B}$ and $q \in X$, one of the geodesics from p to q in X is a curve obtained as a concatenation of a possibly finite subsequence $(C_j)_{j\in\mathbb{N}}$.

The sequence $(C_j)_{j\in\mathbb{N}}$ is called a *parametrization* of X.

The Sierpiński gasket is a piecewise C^1-fractal curve given by a parametrization using the edges of the triangles $\Delta_{n,r}$, for all $n \in \mathbb{N}$ and $r \in \{1,\dots,3^n\}$.

Theorem 3.6.4 ([32, Proposition 2]). *The Sierpiński gasket $\mathscr{SG}_\infty$ is a piecewise C^1-fractal curve, with parametrization $(R_j)_{j\in\mathbb{N}}$ given, for each $j \in \mathbb{N}$, by either of the two affine functions from $[0,1]$ onto*

- *the bottom edge of $\Delta_{n,j}$, if $j = \varkappa(n,r)$ for some $(n,r) \in \Xi$,*
- *the right edge of $\Delta_{n,j}$, if $j = \varkappa(n,r)+1$ for some $(n,r) \in \Xi$,*
- *the left edge of $\Delta_{n,j}$, if $j = \varkappa(n,r)+2$ for some $(n,r) \in \Xi$,*

with

$$\Xi := \left\{(n,r) \in \mathbb{N}^2 : n \in \mathbb{N}, r \in \{0,\dots,3^n-1\}\right\}$$

and $\varkappa(n,r) := 3\left(\sum_{k=0}^{n-1} 3^k + r\right)$, for all $(n,r) \in \Xi$; by convention $\chi(0,0) = 0$.

We now turn to the construction of a metric spectral triple on C^1-fractal curve.

We now define a spectral triple on CP, using the Gelfand–Naimark–Segal representation of CP for the Haar state. Explicitly, let $\mathscr{J}$ be the Hilbert space closure of CP for the inner product

$$(f,g) \in \text{CP} \mapsto \int_{-1}^{1} fg.$$

As usual, we identify $f \in \text{CP}$ with the (bounded) multiplication operator by f on $\mathscr{J}$.

For each $k \in \mathbb{Z}$, let

$$e_k : t \in [-1,1] \mapsto \exp(i\pi k t).$$

Clearly, $e_k \in \mathscr{J}$. We define $\eth$ as the closure of the linear extension of the map defined as follows:

$$\forall k \in \mathbb{Z}, \quad \eth e_k = \pi k e_k.$$

The operator $\eth$ is self-adjoint with spectrum $\{\pi k : k \in \mathbb{Z}\}$. In particular, $\eth$ has a compact resolvent.

A quick computation now shows that $f \in \mathscr{J}$ is in the domain $\mathrm{dom}(\eth)$ of $\eth$ if and only if there exists a necessarily unique $g \in \mathscr{J}$ such that

$$\forall x \in [-1,1], \quad f(x) = f(0) + \int_0^x g(t)\,dt;$$

i.e., f is absolutely continuous on $[-1,1]$, with almost everywhere derivative g. Furthermore, in this case, $\eth f = ig$. From this, it follows that for all $k \in \mathbb{Z}$, we have $[\eth, f]e_k = (\eth f)e_k$. We thus deduce that, if we let

$$\mathsf{L}_{\mathbb{T}} : f \in \mathrm{CP} \mapsto \left|\left|\left|[\eth, f]\right|\right|\right|_{\mathscr{J}} \qquad \text{(allowing for the value } \infty\text{)},$$

then

$$\forall f \in \mathrm{CP}, \quad \mathsf{L}_{\mathbb{T}}(f) = \left\|\eth f\right\|_{L^\infty([-1,1])} \qquad \text{(also allowing for the value } \infty\text{)}.$$

From this, we conclude that $f \in \mathrm{dom}(\mathsf{L}_{\mathbb{T}})$ if and only if $\eth(f)$ is essentially bounded on $[-1,1]$. Equivalently, via the Lebesgue differentiation theorem, $f \in \mathrm{dom}(\mathsf{L}_{\mathbb{T}})$ if and only if f is Lipschitz for the *usual metric* on $[-1,1]$ — with the obvious identification of $\mathscr{J}$ as a closed subspace of $L^2([-1,1])$. In turn, this implies that

$$\left(\mathrm{CP}, \mathscr{J}, \eth\right) \text{ is a metric spectral triple}$$

since $\{f \in \mathrm{CP} : f(0) = 0, \mathsf{L}_{\mathbb{T}}(f) \leqslant 1\}$ is compact in $C([-1,1])$ by Arzéla–Ascoli theorem. However, we want to understand the metric induced by $\mathsf{L}_{\mathbb{T}}$ on the Gelfand spectrum $\mathbb{T}$ of the C*-algebra CP. Let $x, y \in [0,1)$. It is easy to see that

$$\mathsf{mk}_{\mathsf{L}_{\mathbb{T}}}\left(\exp(2i\pi x), \exp(2i\pi y)\right) = \\ \sup\left\{|f(x) - f(y)| : f \in \mathrm{CP}, f(1) = 0, \left\|\eth(f)\right\|_{L^\infty([0,1])} \leqslant 1\right\}.$$

If $f \in \mathrm{CP}$ with $\mathsf{L}_{\mathbb{T}}(f) \leqslant 1$ and $f(1) = 0$, and thus $f(-1) = 0$, then

$$|f(x) - f(y)| \leqslant \min\{|x-y|, 2 - |x-y|\}$$

and therefore,

$$\forall x, y \in [-1,1], \quad \mathsf{mk}_{\mathsf{L}_{\mathbb{T}}}(\exp(2i\pi x), \exp(2i\pi y)) = |x-y| \pmod 1, \tag{3.6.2}$$

where equality in Equation (3.6.2) is achieved by using continuous piecewise affine functions.

Thus, the metric induced by $\mathsf{mk}_{\mathsf{L}_{\mathbb{T}}}$ makes the Gelfand spectrum of CP isometric to the unit circle in $\mathbb{T}$ endowed with its geodesic distance; i.e., the distance between two distinct points is the smallest of the lengths of the two arcs between these points.

We now use the spectral triple $(\mathrm{CP}, \mathscr{J}, \partial)$ in order to construct a spectral triple on the unit interval $[0,1]$. As the construction of spectral triples on piecewise C^1-fractal curves involves possibly countable direct sums of interval spectral triples, we will in particular avoid having the eigenvalue 0 in the spectrum of our interval Dirac operator, so that a countable direct sum of such operators will still have a compact resolvent.

If $f \in C([0,1])$, then the map $t \in [-1,1] \mapsto f(|t|)$ is in CP. Let ϖ be the faithful *-representation of $C([0,1])$ on $\mathscr{J}$ defined by

$$\forall f \in C([0,1]), \quad \forall \xi \in \mathscr{J}, \quad \varpi(f)\xi : t \in [-1,1] \mapsto f(|t|)\xi(t).$$

We also set $\not{D} = \partial + \frac{\pi}{2}$ and $\operatorname{dom}(\not{D}) = \operatorname{dom}(\partial)$, noting that $\not{D}$ is a self-adjoint operator with spectrum $\operatorname{Sp}(\not{D}) = \left\{\pi\left(k+\frac{1}{2}\right) : k \in \mathbb{Z}\right\}$. It is easy to check that $\left(C([0,1]), \mathscr{J}, \not{D}\right)$ is a metric spectral triple over $C([0,1])$ which induces the usual metric on $[0,1]$.

We now will work with the following hypothesis.

Hypothesis 3.6.5. Let $\mathscr{FC}_\infty$ be a piecewise C^1-fractal curve with parametrization $(C_j)_{j\in\mathbb{N}}$; so that, in particular,

$$\mathscr{FC}_\infty = \text{closure of } \bigcup_{j\in\mathbb{N}} \operatorname{range}\left(C_j\right).$$

We denote its geodesic distance by d_∞. We also denote the set of all the endpoints of the curves C_j $(j \in \mathbb{N})$ — which we call the vertices of $\mathscr{FC}_\infty$ — by V_∞.

Let $(B_n)_{n\in\mathbb{N}}$ be an approximation sequence for $\mathscr{FC}_\infty$ adapted to the parametrization $(C_j)_{j\in\mathbb{N}}$, and set $B_\infty = \infty$. For each $n \in \mathbb{N}$, we write

$$\mathscr{FC}_n = \bigcup_{j=0}^{B_n} \operatorname{range}\left(C_j\right).$$

We also denote the geodesic distance on $\mathscr{FC}_n$ by d_n. Last, we let

$$V_n = \left\{C_j(0), C_j(1) : j \in \{0,\ldots,B_n\}\right\},$$

which we call the *set of vertices* of $\mathscr{FC}_n$, and we set $V_\infty = \bigcup_{j\in\mathbb{N}} V_j$ — whose elements we refer to as the *vertices* of $\mathscr{FC}_\infty$.

For every $n \in \overline{\mathbb{N}}$, we denote the Lipschitz seminorm on $\mathscr{FC}_n$ induced by the geodesic distance d_n by L_n.

Finally, for every $j \in \mathbb{N}$, we also denote the length of C_j by λ_j.

For each $n \in \overline{\mathbb{N}}$, we now construct our spectral triple over $\mathscr{FC}_n$, where we use Hypothesis 3.6.5. We let

$$\mathscr{H}_n = \oplus_{j=0}^{B_n} \mathscr{J}$$

and

$$\operatorname{dom}(D_n) = \left\{(\xi_j)_{j=0}^{B_n} \in \mathscr{H}_n : \forall j \in \{0,\ldots,B_n\} \quad \xi_j \in \operatorname{dom}(\not{D})\right\}.$$

For each $j \in \mathbb{N}$, we also let $q_j : C(\mathscr{F}\mathscr{C}_\infty) \twoheadrightarrow C([0,1])$, which sends $f \in C(\mathscr{F}\mathscr{C}_\infty)$ to $f \circ C_j$ in $C[0,1]$. Of course, q_j is a *-epimorphism. We then set, for all $f \in C(\mathscr{F}\mathscr{C}_n)$:

$$\forall \xi = (\xi_j)_{j\in\mathbb{N}, j\leqslant B_n} \in \mathscr{H}_n, \quad \pi_n(f)\xi = \left(\varpi(f \circ C_j)\xi_j\right)_{j\in\mathbb{N}, j\leqslant B_n}.$$

Finally, using the same notation as above, we set:

$$\forall \xi = (\xi_j)_{j\in\mathbb{N}, j\leqslant B_n} \in \mathrm{dom}\,(D_n), \quad D_n\xi = \left(\frac{1}{\lambda_j}\not{D}\xi_j\right)_{j\in\mathbb{N}, j\leqslant B_n},$$

where λ_j is the length of C_j, for every $j \in \mathbb{N}$.

It is then easily checked that $(C(\mathscr{F}\mathscr{C}_n), \mathscr{H}_n, D_n)$ is a spectral triple on $C(\mathscr{F}\mathscr{C}_n)$. We will next show that this spectral triple is metric and that mk_{D_n} restricted to $\mathscr{F}\mathscr{C}_n$ coincides with the geodesic distance d_n. Our theorem includes [9, Theorem 8.13] (case $n = \infty$) and extends it to all $n \in \overline{\mathbb{N}}$, which we need in order to be able to formulate and establish our approximation results.

Theorem 3.6.6. *We assume Hypothesis 3.6.5. Let $n \in \overline{\mathbb{N}}$. If $f \in \mathscr{F}\mathscr{C}_n$, then*

$$f \in \mathrm{dom}\,(\mathsf{L}_n) \iff f\,\mathrm{dom}\,(D_n) \subseteq \mathrm{dom}\,(D_n)$$

and, for all $f \in \mathrm{dom}\,(\mathsf{L}_n)$, we have

$$\mathsf{L}_n(f) = \left|\!\left|\!\left| [D_n, \pi_n(f)] \right|\!\right|\!\right|_{\mathscr{H}_n}.$$

In particular, the restriction of mk_{D_n} to $\mathscr{F}\mathscr{C}_n$ is the geodesic distance d_n.

Remark 3.6.7. It is not sufficient to show that the restriction of mk_{D_n} is d_n in order to conclude that Theorem 3.6.6 holds — see, for instance, [2], where two different L-seminorms on the continuous functions over the Cantor set give the same metric on the Cantor set but *not* on the state space.

Theorem 3.6.8. *If Hypothesis 3.6.5 holds, then*

$$\lim_{n\to\infty} \Lambda^{\mathsf{spec}}\left((C(\mathscr{F}\mathscr{C}_n), \mathscr{H}_n, D_n), (C(\mathscr{F}\mathscr{C}_\infty), \mathscr{H}_\infty, D_\infty)\right) = 0.$$

Bibliography

[1] K. Aguilar and J. Kaad, *The podleś sphere as a spectral metric space.*, J. Geom. Phys. **133** (2018), 260–278.

[2] K. Aguilar and F. Latrémolière, *Quantum ultrametrics on AF algebras and the Gromov–Hausdorff propinquity*, Studia Math. **231** (2015), no. 2, 149–194, ArXiv: 1511.07114.

[3] ______, *Some applications of conditional expectations to convergence for the Gromov-Hausdorff propinquity*, Banach Center Publ. **120** (2021), 35–46, arXiv:1708.00595.

[4] K. Aguilar, F. Latrémolière, and T. Rainone, *Bunce-deddens algebras as quantum gromov-hausorff distance limits of circle algebras*, Integral Equations Operator Theory **94** (2022), no. 1, Paper no. 2, 42pp, ArXiv: 2008.07676.

[5] C. Antonescu and E. Christensen, *Spectral triples for af C^*-algebras and metrics on the cantor set*, J. Oper. Theory **56** (2006), no. 1, 17–46, arXiv: 0309044.

[6] H. Brézis, *Analyse fonctionnelle*, Collection Mathématiques Appliquées Pour la Maitrise, Masson, Paris, 1983.

[7] F. Latrémolière C. Farsi, J. Packer, *Convergence of inductive sequences of spectral triples for the spectral propinquity*, Submitted (2023), 54 pages., arXiv:2301.00274.

[8] M. Christ and M. A. Rieffel, *Nilpotent group C^*-algebras-algebras as compact quantum metric spaces*, Canad. Math. Bull. **60** (2017), no. 1, 77–94, arXiv: 1508.00980.

[9] E. Christensen, C. Ivan, and M. Lapidus, *Dirac operators and spectral triples for some fractal sets built on curves*, Adv. Math. **217** (2008), no. 1, 42–78.

[10] A. Connes, *C*–algèbres et géométrie differentielle*, C. R. de l'Academie des Sciences de Paris (1980), no. series A-B, 290.

[11] A. Connes, *Compact metric spaces, Fredholm modules and hyperfiniteness*, Ergodic Theory Dynam. Systems **9** (1989), no. 2, 207–220.

[12] A. Connes and M. A. Rieffel, *Yang-mills for noncommutative two-tori*, Contemporary Math **62** (1987), no. Operator algebras and mathematical physics (Iowa City, Iowa, 1985), 237–266.

[13] R. L. Dobrushin, *Prescribing a system of random variables by conditional probabilities*, Theory Probab. Appl. **15** (1970), no. 3, 459–486.

[14] R. M. Dudley, *Real analysis and probability*, 2002 ed., Cambridge Studies in Advanced Mathematics, vol. 74, Cambridge University Press, 2002.

[15] David A. Edwards, *The structure of superspace*, Studies in topology (Proc. Conf., Univ. North Carolina, Charlotte, N. C., 1974; dedicated to Math. Sect. Polish Acad. Sci.), 1975, p. 121–133. MR 0401069

[16] E. G. Effros, D. E. Handelman, and C. L. Shen, *Dimension groups and their affine representations*, Amer. J. Math. **102** (1980), no. 2, 385–407.

[17] E. G. Effros and C. L. Shen, *Approximately finite C^*-algebras and continued fractions*, Indiana Univ. Math. J. **29** (1980), no. 2, 191–204.

[18] R. Floricel and A. Ghorbanpour, *On inductive limit spectral triples*, Proc. Amer. Math. Soc. **147** (2019), no. 8, 3611–3619, ArXiv: 1712.09621.

[19] M. Gromov, *Groups of polynomial growth and expanding maps*, Publ. Math. Inst. Hautes Études Sci. **53** (1981), 53–78.

[20] ______, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, Birkhäuser, 1999.

[21] F. Hausdorff, *Grundzüge der Mengenlehre*, Verlag Von Veit und Comp., 1914.

[22] A. Hawkins, A. Skalski, S. White, and J. Zacharias, *On spectral triples on crossed products arising from equicontinuous actions*, Math. Scand. **113** (2013), no. 2, 262–291.

[23] A. Hawkins, A. Skalski, S White, and J. Zacharias, *On spectral triples on crossed products arising from equicontinuous actions*, Math. Scand. **113** (2013), 262–291, arXiv:1103.6199.

[24] R. Hoegh-Krohn, M. B. Landstad, and E. Stormer, *Compact ergodic groups of automorphisms*, Ann. of Math. **114** (1981), 75–86.

[25] C. Farsi F. Latremoliere J. Bassi, R. Conti, *Isomertry groups of inductive limits of metric spectral triples and gromov-hausdorff convergence*, Submitted (2023), 38 pages, arXiv:2302.09117.

[26] R. Kadison and J. Ringrose, *Fundamentals of the theory of operator algebras I*, Graduate Studies in Mathematics, vol. 15, AMS, 1997.

[27] L. V. Kantorovich, *On one effective method of solving certain classes of extremal problems*, Dokl. Akad. Nauk. USSR **28** (1940), 212–215.

[28] L. V. Kantorovich and G. Sh. Rubinstein, *On the space of completely additive functions*, Vestnik Leningrad Univ., Ser. Mat. Mekh. i Astron. **13** (1958), no. 7, 52–59, In Russian.

[29] D. Kerr, *Matricial quantum Gromov-Hausdorff distance*, J. Funct. Anal. **205** (2003), no. 1, 132–167, math.OA/0207282.

[30] M. Klisse, *Crossed products as compact quantum metric spaces*, Submitted (2023).

[31] T. Landry, M. Lapidus, and F. Latrémolière, *Metric approximations of the spectral triple on the Sierpinki gasket and other fractals*, Adv. Math. **385** (2021), Paper No. 107771, 43 pp.

[32] M. Lapidus and J. Sarhad, *Dirac operators and geodesic metric on the harmonic sierpinski gasket and other fractal sets*, J. Noncommut. Geom. **8** (2014), no. 4, 947–985.

[33] F. Latrémolière, *Continuity of the spectrum of dirac operators of spectral triples for the spectral propinquity*, Submitted (2022), 49 pages.

[34] F. Latrémolière, *Convergence of fuzzy tori and quantum tori for the quantum Gromov–Hausdorff propinquity: an explicit approach.*, Münster J. Math. **8** (2015), no. 1, 57–98, arXiv: math/1312.0069.

[35] ______, *Curved noncommutative tori as Leibniz compact quantum metric spaces*, J. Math. Phys. **56** (2015), no. 12, 123503, 16 pages, arXiv: 1507.08771.

[36] ______, *The dual Gromov–Hausdorff propinquity*, J. Math. Pures Appl. **103** (2015), no. 2, 303–351, arXiv: 1311.0104.

[37] ______, *Equivalence of quantum metrics with a common domain*, J. Math. Anal. Appl. **443** (2016), 1179–1195, arXiv: 1604.00755.

[38] ______, *The quantum Gromov-Hausdorff propinquity*, Trans. Amer. Math. Soc. **368** (2016), no. 1, 365–411.

[39] ______, *A compactness theorem for the dual Gromov-Hausdorff propinquity*, Indiana Univ. Math. J. **66** (2017), no. 5, 1707–1753, arXiv: 1501.06121.

[40] ______, *Convergence of Cauchy sequences for the covariant Gromov-Hausdorff propinquity*, J. Math. Anal. Appl. **469** (2019), no. 1, 378–404, arXiv: 1806.04721.

[41] ______, *The modular Gromov–Hausdorff propinquity*, Dissertationes Math. **544** (2019), 1–70, arXiv: 1608.04881.

[42] ______, *Convergence of Heisenberg modules for the modular Gromov-Hausdorff propinquity*, J. Operator Theory **84** (2020), no. 1, 211–237.

[43] ______, *The covariant Gromov-Hausdorff propinquity*, Studia Math. **251** (2020), no. 2, 135–169, arXiv: 1805.11229.

[44] ______, *Heisenberg modules over quantum* 2*-tori are metrized quantum vector bundles*, Canad. J. Math **72** (2020), no. 4, 1044–1081, arXiv: 1703.07073.

[45] ______, *Convergence of spectral triples on fuzzy tori to spectral triples on quantum tori*, Comm. Math. Phys. **388** (2021), no. 2, 1049–1128, arXiv: 2102.03729.

[46] ______, *The dual-modular Gromov-Hausdorff propinquity and completeness*, J. Noncomm. Geometry **115** (2021), no. 1, 347–398.

[47] ______, *The Gromov-Hausdorff propinquity for metric spectral triples*, Adv. Math. **404** (2022), Paper No. 108393, 56.

[48] F. Latrémolière and J. Packer, *Noncommutative solenoids and the Gromov-Hausdorff propinquity*, Proc. Amer. Math. Soc. **145** (2017), no. 5, 1179–1195, arXiv: 1601.02707.

[49] H. Li, *C^*-algebraic quantum Gromov-Hausdorff distance*, (2003), arXiv: math.OA/0312003.

[50] E. J. McShane, *Extension of range of functions*, Bull. Amer. Math. Soc. **40** (1934), no. 12, 825–920.

[51] N. Ozawa and M. A. Rieffel, *Hyperbolic group C^*-algebras and free product C^*-algebras as compact quantum metric spaces*, Canad. J. of Math. **57** (2005), 1056–1079, arXiv: math/0302310.

[52] M. A. Rieffel, *Metrics on states from actions of compact groups*, Doc. Math. **3** (1998), 215–229, math.OA/9807084.

[53] ______, *Metrics on state spaces*, Doc. Math. **4** (1999), 559–600, math.OA/9906151.

[54] ______, *Gromov-Hausdorff distance for quantum metric spaces*, Mem. Amer. Math. Soc. **168** (2004), no. 796, 1–65, math.OA/0011063.

[55] ______, *Distances between matrix alegbras that converge to coadjoint orbits*, Proc. Sympos. Pure Math. **81** (2010), 173–180, arXiv: 0910.1968.

[56] ______, *Leibniz seminorms for "matrix algebras converge to the sphere"*, Clay Mathematics Proceedings **11** (2010), 543–578, arXiv: 0707.3229.

[57] Marc A. Rieffel, *Matricial bridges for "matrix algebras converge to the sphere"*, Operator algebras and their applications, Contemp. Math., vol. 671, Amer. Math. Soc., Providence, RI, 2016, arXiv: 1502.00329, p. 209–233. MR 3546687

[58] L. N. Wasserstein, *Markov processes on a countable product space, describing large systems of automata*, Problemy Peredachi Infomatsii **5** (1969), no. 3, 64–73, In Russian.

[59] J. Wheeler, *Superspace and the nature of quantum geometrodynamics*, Battelle Rencontres. 1967 Lectures in Mathematics and Physics (DeWitt C. and J. Wheeler, eds.), 1968.

[60] W. Wu, *Non-commutative metric topology on matrix state spaces*, Proc. Amer. Math. Soc. **134** (2006), no. 2, 443–453, arXiv: math.OA/0410587.

[61] ______, *Quantized Gromov-Hausdorff distance*, J. Funct. Anal. **238** (2006), no. 1, 58–98, arXiv: math.OA/0503344.